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RESEARCH ARTICLE

On the Boundedness Properties of Mikhlin Operator on Generalized Morrey Spaces

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Abstract:
In this paper we investigate the boundedness of Mikhlin perators on generalized Morrey spaces. The results show that the operators are bounded on generalized Morrey spaces under some assumptions.

Keywords: Mikhlin Operator, Generalized Morrey Spaces

1. Introduction

The concept of differential equation has many application in real life. It can be used to model some pyschical problems, such as heat diffusion and wave equation. One of the model for heat diffusion in the form of partial differential equation is the following system:

$$\begin{aligned} u_t &= ku_{xx}, \quad x \in \mathbb{R}, t > 0, \\ u(x,0) &= \phi(x), \quad x \in \mathbb{R} \end{aligned}$$

where k is a positive constant and $u(x,t)$ represents the temperature of a wire at a point x and at the time $t \geq 0$, and the function $\phi(x)$ is the initial condition, namely the temperature of the wire at point x and at the time $t = 0$.

One may solve the system and obtain that the solution is given by

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)\phi(y)dy$$

where K is a kernel defined by

$$K(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{(x-y)^2}{4kt}}$$

Moreover, we can write the solution as a convolution

$$u(x,t) = (K(\cdot,t) * \phi)(x)$$

Fourier transform $\mathcal{F}$ then implies that

$$\mathcal{F}(u) = \hat{u} = (K \hat{*} \phi) = (2\pi)^{n/2}\hat{K}\hat{\phi}$$

We see that $\hat{\phi}$ is mapped to $\hat{u}(x,t)$ by "multiplying" with $\hat{K}$. The operator is called multiplier operator generated by $\hat{K}$ or Fourier multiplier.

S.G. Mikhlin generalized the Fourier multiplier operator in 1956. The operator was then called Mikhlin operator. He also prove that the operator is bounded on Lebesgue space $L^p(\mathbb{R}^n) = L^p$ for $1 < p < \infty$. There are some generalization of Lebesgue spaces. One of the important generalization is Morrey space.

Morrey space was first introduced by Morrey [1] in 1938 that used to study the local behavior of solution for elliptic differential equation. Related to Mikhlin Operator, Maharani *et al* [2] had proved that the operator was bounded on Morrey space which exactly extends the result by Mikhlin.

Generalized Morrey space was then introduced as in [3] as the generalization of the (classical) Morrey space. Mizuhara proved the boundedness of some singular integral operators on the space. In this article, we investigate the assumption that ensure the boundedness of the Mikhlin operator on Generalized Morrey space as in last section.

2. Definitions and Previous Results

In this section, we provide some previous results and definitions. We start with the notion of multi-index. Multi-index α is ordered $n-$tuple of non-negative integer, namely

$$\alpha = (\alpha_1, \cdots, \alpha_n)$$

where $\alpha_1, \cdots, \alpha_n \in \mathbb{N} \cup \{0\}$. Moreover, for multi-index α, the symbol $|\alpha|_*$ represents the size of α that is defined as

$$|\alpha|_* = \alpha_1 + \cdots + \alpha_n,$$

and for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

The following lemma contains some basic properties of multi-index.

Lemma 2.1. *If α is multi-index and $k \in \mathbb{N}$, then there is a constant $C_{n,\alpha} > 0$ and $C_{n,k} > 0$ such that*

$$|x^\alpha| \leq C_{n,\alpha} |x|^{|\alpha|_*},$$

and

$$|x|^k \leq C_{n,k} \sum_{|\beta|_* = k} |x^\beta|$$

for all $x \in \mathbb{R}^n$. □

The proof of lemma 2.1 are elementary and we omit it here. Next, we provide the definition and properties of Schwartz space and tempered distribution space. These spaces play important roles in Fourier analysis and harmonic analysis. The followings are the definition and characterization of Schwartz space.

Definition 2.1. [4] Schwartz space, denoted by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, is set of all functions f such that for all multi-indexes α and β, there is a positive constant $C_{\alpha,\beta}$ satisfying

$$\rho_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha,\beta} < \infty$$

By using properties of multi-indexes, one may prove the following lemma of characterization of Schwartz function. The lemma tells us that the Schwartz function is infintely differentiable function that is rapidly decreasing [5, 6].

Lemma 2.2. *$f \in \mathcal{S}$ if and only if for all $N \in \mathbb{N}$ and multi-indexes α, there is a constant $C_{\alpha,\beta>0}$ such that*

$$|(\partial^\alpha f)(x)| \leq C_{\alpha,N}(1+|x|)^{-N}$$

for all $x \in \mathbb{R}^n$. □

For the discussion about the topology of Schwartz space, one may refer to [4]. Next, we define the the tempered distribution space.

Definition 2.2. [4] Tempered distribution space, denoted by $\mathcal{S}'$, is set of all bounded linear functional on $\mathcal{S}$.

The following lemma gives us a class of tempered distribution.

Lemma 2.3. *[7] Let f be a function on $\mathbb{R}^n$ such that for some $N \in \mathbb{N}$ we have*

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|x|)^N} dy < \infty.$$

Then, f is a tempered distribution. □

Next, we provide the definition and properties of Mikhlin Operator, particularly in Lebesgue spaces.

Definition 2.3. [8] Let $N = n+2$ where $n \in \mathbb{N}$ and $m : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{C}$ be an $N-$times differentiable function such that

$$|\partial_x^\alpha m(x)| \leq C|x|^{-|\alpha|_*}$$

for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for multi-index α where $|\alpha|_* \leq N$ where C is positive constant that does not depend on x and α. A Mikhlin operator is defined as

$$M(f) = \mathcal{F}[m\hat{f}]$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2.1. *[8] Let M be a Mikhlin operator. Then, M can be extendend to a bounded linear operator in L^p for $1 < p < \infty$.* □

By theorem 2.1 and properties of Fourier transform, we may rewrite the Mikhlin Operator as follows.

$$M(f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy = (K * f)(x), \quad x \in \mathbb{R}^n$$

for suitable function f where K is a kernel with properties as in the following theorem.

Theorem 2.2. *[8] Let M be a Mikhlin operator. Then there is a locally integrable and continuously differentiable funtion $K : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{C}$ with compact support which satisfies*

$$|K(z)| \leq C\frac{1}{|z|^n}, \quad |\nabla K(z)| \leq C\frac{1}{|z|^{n+1}}$$

for $z \neq 0$ where $C > 0$ is independent of z and

$$M(f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

for all $x \notin \operatorname{supp}(f)$, $f \in L^2(\mathbb{R}^n)$. □

We know that the Fourier transform is extended to any function in tempered distribution space. Then,

$$M(f)(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

for all $x \notin \text{supp}(f)$ where f is a function in tempered distribution.

We end this section with the definition of Generalized Morrey space.

Definition 2.4. (Generalized Morrey Space) Let Ω be a measurable subset of $\mathbb{R}^n$, $1 < p < \infty$ and ψ be a positively function on $\mathbb{R}^n \times (0,\infty)$. The Generalized Morrey space $\mathcal{M}^p_\psi(\Omega)$ is set of all measurable function f such that $\|f\|_{\mathcal{M}^p_\psi} < \infty$ where

$$\|f\|_{\mathcal{M}^p_\psi(\Omega)} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \|f\|_{L^p(B(a,r)\cap\Omega)} = \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi(a,r)} \left(\int_{B(a,r)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

If $\Omega = \mathbb{R}^n$, we write $\mathcal{M}^p_\psi = \mathcal{M}^p_\psi(\Omega)$. In the definition, if we set $\psi(a,r) = r^{\lambda/p}$ where $0 \le \lambda < n$, we have $\mathcal{M}^p_\psi$ as (classical) Morrey space. Moreover, if we set ψ as a positively constant function, $\mathcal{M}^p_\psi$ is Lebesgue space L^p. If we can prove the boundedness of Mihklin operator on Generalized Morrey space, we also have the same property for (classical) Morrey space and Lebesgue space L^p.

3. Boundedness of Mikhlin Operator on Generalized Morrey Spaces

In this section, we state our main result and its proof. Our main results are in the following theorem.

Theorem 3.1. *Let M be a Mikhlin operator. Suppose the functions ψ_1 and ψ_2 on $\mathbb{R}^n \times (0,+\infty)$ satisfy that for some $C > 0$,*

$$\sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \inf_{s<t<\infty} \psi_1(a,t) \frac{ds}{s} \le C.$$

Then, M can be extended to be a bounded linear operator from $\mathcal{M}^p_{\psi_1}$ to $\mathcal{M}^p_{\psi_2}$ for $1 < p < +\infty$.

By elementary facts in real analysis, we have the following corollary.

Corollary 3.1. *Let M be a Mikhlin operator. Suppose the functions ψ_1 and ψ_2 on $\mathbb{R}^n \times (0,+\infty)$ satisfy that for some $C > 0$,*

$$\sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \operatorname*{ess\,inf}_{s<t<\infty} \psi_1(a,t) \frac{ds}{s} \le C.$$

Then, M can be extended to be a bounded linear operator from $\mathcal{M}^p_{\psi_1}$ to $\mathcal{M}^p_{\psi_2}$ for $1 < p < +\infty$.

Corollary 3.2. *Let M be a Mikhlin operator. Suppose the functions ψ_1 and ψ_2 on $\mathbb{R}^n \times (0,+\infty)$ satisfy that for some $C > 0$,*

$$\sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \psi_1(a,s) \frac{ds}{s} \le C.$$

Then, M can be extended to be a bounded linear operator from $\mathcal{M}^p_{\psi_1}$ to $\mathcal{M}^p_{\psi_2}$ for $1 < p < +\infty$.

Before proving Theorem 3.1, we first provide the following theorems.

Theorem 3.2. *Suppose that p, ψ_1, and ψ_2 as in Theorem 3.1. Let $f \in \mathcal{M}^p_{\psi_1}$ and $B(a,r)$ a ball in $\mathbb{R}^n$. Then,*

$$\int_{(2B(a,r))^c} \frac{|f(y)|}{|a-y|^n} dy \le C \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{p}}} \|f\|_{L^p(B(a,s))} \frac{ds}{s}$$

where $C > 0$ is independent of f, a, and r.

Proof. By using Fubini's theorem and Holder's inequality,

$$\begin{aligned}
\int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^n} dy &\le C \int_{2r}^{\infty} \frac{1}{|B(a,s)|} \int_{B(a,s)} |f(y)| dy \frac{ds}{s} \\
&\le C \int_{2r}^{\infty} \frac{1}{|B(a,s)|} \|f\|_{L^p(B(a,s))} \|1\|_{L^{p'}(B(a,s))} \frac{ds}{s} \\
&\le C \int_{2r}^{\infty} \frac{|B(a,s)|^{1/p'}}{|B(a,s)|} \|f\|_{L^p(B(a,s))} \frac{ds}{s} \\
&= C \int_{2r}^{\infty} \frac{1}{|B(a,s)|^{\frac{1}{p}}} \|f\|_{L^p(B(a,s))} \frac{ds}{s} \le C \int_{r}^{\infty} \frac{1}{|B(a,s)|^{\frac{1}{p}}} \|f\|_{L^p(B(a,s))} \frac{ds}{s}
\end{aligned}$$

It proves the theorem. □

Theorem 3.3. *Suppose that ψ_1 and ψ_2 as in Theorem 3.1. Let $f \in \mathcal{M}_{\psi_1}$ and $B(a,r)$ is a ball in $\mathbb{R}^n$. Then, $f_2 = f \cdot \mathcal{X}_{(2B)^c}$ is tempered distribution.*

Proof. Let $g \in \mathcal{S}$. We see that by Holder's inequality, Theorem 3.2, and Lemma 2.2 (taking $|\alpha|_* = 0$ and $N \in \mathbb{N}$ with $N > n$), the following estimate holds.

$$\begin{aligned}
|\langle f_2, g\rangle| &\le \int_{\mathbb{R}^n} |f_2(x)| \cdot |g(x)| dx = \int_{B(a,2r)^c} |f(x)| \cdot |g(x)| dx \le C \int_{B(a,2r)^c} \frac{|f(y)|}{(1+|y|)^N} dy \\
&= C \int_{B(0,2r)^c} \frac{|f(y)|}{(1+|y|)^N} + C \int_{B(0,2r)} \frac{|f(y)|}{(1+|y|)^N} - C \int_{B(a,2r)} \frac{|f(y)|}{(1+|y|)^N} \\
&\le C \int_{B(0,2r)^c} \frac{|f(y)|}{(1+|y|)^n} dy + C \left[|B(0,2r)| \cdot \max_{y \in B(0,2r)} \frac{1}{(1+|y|)^{Np'}} \right]^{\frac{1}{p'}} \|f\|_{L^p(B(0,2r))} \\
&\quad - C \left[|B(0,2r)| \cdot \min_{y \in B(0,2r)} \frac{1}{(1+|y|)^{Np'}} \right]^{\frac{1}{p'}} \|f\|_{L^p(B(0,2r))} \\
&\le C \int_{B(0,2r)^c} \frac{|f(y)|}{|y|^n} dy + \left[|B(0,2r)| \cdot \max_{y \in B(0,2r)} \frac{1}{(1+|y|)^{Np'}} \right]^{\frac{1}{p'}} \|f\|_{L^p(B(0,2r))} \\
&\le C \int_{2r}^{\infty} \frac{1}{|B(0,s)|^{\frac{1}{p}}} \|f\|_{L^p(B(0,s))} \frac{ds}{s} + C_{r,p} \\
&\le C \frac{1}{r^n} \int_{2r}^{\infty} \|f\|_{L^p(B(0,s))} \frac{ds}{s} + C_{r,p}
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle f_2, g\rangle| &\le C \frac{1}{r^n} \int_{2r}^{\infty} \|f\|_{L^p(B(0,s))} \frac{\sup_{s<t<\infty} \frac{1}{\psi_1(0,t)}}{\sup_{s<t<\infty} \frac{1}{\psi_1(0,t)}} \frac{ds}{s} + C_{r,p} \\
&= C \frac{1}{r^n} \int_{2r}^{\infty} \inf_{s<t<\infty} \psi_1(0,t) \|f\|_{L^p(B(0,s))} \sup_{s<t<\infty} \frac{1}{\psi_1(0,t)} \frac{ds}{s} + C_{r,p} \\
&\le C \frac{1}{r^n} \int_{2r}^{\infty} \inf_{s<t<\infty} \psi_1(0,t) \frac{dt}{t} \cdot \sup_{t>0} \|f\|_{L^p(B(0,t))} \sup_{t<s<\infty} \frac{1}{\psi_1(0,s)} + C_{r,p} \\
&= C \frac{1}{r^n} \int_{2r}^{\infty} \inf_{s<t<\infty} \psi_1(0,t) \frac{dt}{t} \cdot \sup_{t>0} \|f\|_{L^p(B(0,t))} \frac{1}{\psi_1(0,t)} + C_{r,p} \\
&\le C \|f\|_{\mathcal{M}^p_{\psi_1}} \cdot \frac{\psi_2(0,2r)}{r^n} + C_{r,p} \\
&< \infty
\end{aligned}$$

It proves Theorem 3.3 □

Theorem 3.4. *Let M be a Mikhlin operator. Then, for $1 < p < \infty$ we have*

$$\|Mf\|_{L^pB(a,r)} \leq \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s}, \quad (a,r) \in \mathbb{R} \times (0,\infty).$$

Proof. For the function f, we write $f = f_1 + f_2$ where $f_1 = f \cdot \mathcal{X}_{B(a,2r)}$. By the boundedness of M on L^p as in Theorem 2.1, we have

$$\|Mf_1\|_{L^p(B(a,r))} \leq \|Mf_1\|_{L^p} \leq \|f_1\|_{L^p} = \|f\|_{L^p(B(a,2r))}$$

The fact that $t \mapsto \|f\|_{L^p(B(a,t))}$ is increasing then implies

$$\|f\|_{L^p(B(a,r))} \leq \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s}, \quad (a,r) \in \mathbb{R}^n \times (0,+\infty),$$

and

$$\|Mf_1\|_{L^p(B(a,r))} \leq \int_{2r}^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s} \leq \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s}.$$

For f_2, by Theorem 3.2 we have that for $x \in B(a,r)$

$$\begin{aligned} |Mf_2(x)| &\leq \int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x-y|^n}dy \\ &= \int_{(2B)^c} \frac{|f(y)|}{|x-y|^n}dy \\ &\leq \int_{B(a,2r)^c} \frac{|f(y)|}{|a-y|^n}dy \\ &\leq \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{p}}}\|f\|_{L^p(B(a,s))}\frac{ds}{s} \end{aligned}$$

Hence,

$$\|Mf_2\|_{L^pB(a,r)} \leq |B(a,r)|^{\frac{1}{p}} \int_r^\infty \frac{1}{|B(a,s)|^{\frac{1}{p}}}\|f\|_{L^p(B(a,s))}\frac{ds}{s} \leq \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s}$$

Therefore,

$$\|Mf\|_{L^p(B(a,r))} \leq \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s}$$

and this completes the proof. □

Proof of Theorem 3.1. Let $f \in \mathcal{M}^p_{\psi_1}$. By Theorem 3.4, we have

$$\begin{aligned} \|Mf\|_{\mathcal{M}^p_{\psi_2}} &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)}\|f\|_{L^p(B(a,r))} \\ &\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{ds}{s} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \|f\|_{L^p(B(a,s))}\frac{\sup_{s<t<\infty}\frac{1}{\psi_1(a,t)}}{\sup_{s<t<\infty}\frac{1}{\psi_1(a,t)}}\frac{ds}{s} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \inf_{s<t<\infty}\psi_1(a,t)\|f\|_{L^p(B(a,s))}\sup_{s<t<\infty}\frac{1}{\psi_1(a,t)}\frac{ds}{s} \\ &\leq \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \inf_{s<t<\infty}\psi_1(a,t)\frac{dt}{t}\cdot\sup_{t>0}\|f\|_{L^p(B(a,t))}\sup_{t<s<\infty}\frac{1}{\psi_1(a,s)} \\ &= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\psi_2(a,r)} \int_r^\infty \inf_{s<t<\infty}\psi_1(a,t)\frac{dt}{t}\cdot\sup_{t>0}\|f\|_{L^p(B(a,t))}\frac{1}{\psi_1(a,t)} \\ &\leq \|f\|_{\mathcal{M}^p_{\psi_1}} \end{aligned}$$

It completes the proof of theorem 3.1. □

The results on Theorem 3.1 tells us that the Mihklin operator initially defined on L^2 can be extended to generalized Morrey space. Then, the results directly holds on (classical) Morrey space [2] and Lebesgue space 2.1. One may interest to extend this result to some more general function spaces such as variable exponent generalized Morrey space by refering to results in [9].

4. Conclusion

According to the results, it can be concluded that the Mikhlin operator that is initially defined on L^2 can be extended to Generalized Morrey spaces and is bounded from one generalized Morrey space to another generalized Morrey space under some assumptions.

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