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RESEARCH ARTICLE

# The Topological Indices of Coprime Graph for Integer Module Group

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**Abstract**:
This study delves into the fascinating realm of coprime graphs within the integer modulo group, revealing a fundamental property where these graphs manifest as star graphs when the order corresponds to a prime power. This pivotal insight, as elucidated by [1], plays a crucial role in the determination of topological indices for these graphs. The research culminates in the formulation of formulas for the First Zagreb index and the Second index of coprime graphs in the context of integer modulo groups, unveiled in the last two theorems. These numerical topological indices are poised to unveil intricate correlations among group elements, contingent upon their order, echoing the foundational principles of chemical graph theory. This exploration not only enriches our understanding of group structu.

**Keywords:** Coprime Graph, the First Zagreb Index, the Second Zagreb Index

## 1. Introduction

Chemical Graph Theory is a scientific discipline that employs mathematical graphs to comprehend and anticipate the properties of molecules and chemical compounds. In this cotext, molecules are represented graphically, with atoms serving as vertices and chemical bonds as edges [2]. The primary aim is to scrutinize molecular structures, assess properties, and forecast chemical behaviors based on connectivity patterns. This field utilizes graph theory concepts to advance research in drug development, the modeling of chemical properties, and the design of molecules with specific characteristics. Consequently, Chemical Graph Theory plays an integral role in diverse facets of modern chemistry [3].

Within the realm of algebraic group theory, there is a profusion of investigations concerning the characteristics and numerical invariants of various graph representations across different groups [4]. [5] has taken on modeling the power graph within the dihedral group, while [6] has explored the power graph within the integer modulo group. The representation and attributes of intersection graphs were then established by Nurhabibah in the dihedral group [7] and subsequently refined by

[8]. Furthermore, contributed graph representation models in the dihedral group, which were subsequently employed to construct graph representation models in the generalized quaternion group by [9]. They have also successfully introduced a dual representation within the non-coprime group [10], with a version in the dihedral group offered by Misuki [11] and further generalized by Aulia [12]. In addition, the domain of ring structures has witnessed extensive exploration, with scholars delving into the characterization and numerical invariants of various graphs, including the zero divisor graph, prime graph, and nilpotent graph [13], [14], [15], [16], [17], [18], [19].

Building upon the characterizations initially developed for the coprime graph, researchers have embarked on the task of formulating topological indices. Notably, there have been significant studies that focus on specific groups, such as the integer modulo group, dihedral group, and generalized quaternion group. Juliana's research primarily revolves around coprime groups within the integer modulo group, while Syarifudin is dedicated to exploring coprime groups within the dihedral group [20], and Nurhabibah is actively investigating generalized quaternion groups [21]. Their collective efforts revolve around characterizing these graphs and illuminating their numerical invariants.

For example, within the context of the dihedral group, Yatin has provided formulas for the hyper-Wiener index and the Padmakar-Ivan index [17], while Gayatri has contributed formulas for the Harary index, Harmonic index, Zagreb index, Gutman index, and Wiener index [18].

In contrast, research into topological indices within the integer modulo group has been comparatively limited, with only Husni offering formulas for the Harmonic index and Gutman index [22], and Devandra addressing the Harary index [23]. To bridge this gap, we present the formula for the Zagreb index in the integer modulo group, thereby completing the list of topological indices. This comprehensive overview underscores the dynamic and evolving nature of Chemical Graph Theory, shedding light on the intricate relationships between mathematical groups, graph theory, and chemical compounds.

In this study, we exclusively consider simple graphs, which are undirected and had no cycle. The following is a list of basic terms we use in this text.

**Definition 1.1.** *The number of edges that are incident to a vertex is known as the graph's degree. It is denotated as* $\deg(g)$ *for any vertex* $g$.

**Definition 1.2.** *For any two vertices* $u$ *and* $v$ *in a graph* $G$*, the distance between* $u$ *and* $v$ *is defined to be the length of the shortest path between* $u$ *and* $v$*, denoted* $d(u, v)$.

**Definition 1.3.** If $G$ is a group with identity $e$ and $g \in G$, the order ofg is the smallest natural number $k$ such that $g^k = e$ and we write $|g| = k$.

A coprime graph, often referred to as a "relatively prime graph", represents a unique mathematical concept where the vertices correspond to positive integers, and two nodes share an edge if and only if their corresponding integers are coprime, meaning they have no common factors other than 1.

**Definition 1.4.** *When two different vertices,* $a$ *and* $b$*, are neighboring exclusively if and only if* $(|a|, |b|) = 1$*, then the graph is called the coprime graph of group* $G$*, or* $\Gamma_G$.

The first Zagreb index is a topological index in graph theory that quantifies the degree-based properties of a graph by summing the squared degrees of its vertices.

**Definition 1.5.** *Let* $G$ *be a graph, The First Zagreb Index* $(M_1(G))$ *from graph* $G$ *is*

$$M_1(G) = \sum_{v_i \in V(G)} \deg(v_i)^2$$

*where* $\deg(v_i)$ *is the degree of vertex* $v_i$.

**Definition 1.6.** *Let G be a graph, The Second Zagreb index $(M_2(G))$ from graph G is*

$$M_2(G) = \sum_{v_i v_2 \in E(G)} \deg(v_1)\deg(v_2)$$

*where* $\deg(v_i)$ *is the degree of vertex* $v_i$.

## 2. Main Results

Syarifudin presents interesting properties related to the shape of the coprime graph for integer modulo groups in the following Theorem.

**Theorem 2.1.** *If $n = p^k$ is for some prime p and positive integer, then the coprime graph of $\mathbb{Z}_n$ is a star graph, with the identity as the center.*

**Example 2.1.** According to Theorem 2.1, the coprime graph of group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is a star graph.

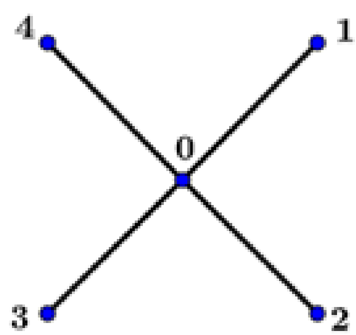


Figure 2.1: Coprime Graph $\mathbb{Z}_5$

Therefore, the identity's degree is 4, and a vertex'sother degree is 1. Thus, the following is the First Zagreb index of $\Gamma_{\mathbb{Z}_5}$.

$$\begin{aligned} M_1(\Gamma_{\mathbb{Z}_5}) &= \sum_{v_i \in V(\Gamma_{\mathbb{Z}_5})} \deg(v_i)^2 \\ &= \deg(0)^2 + \deg(1)^2 + \deg(2)^2 + \deg(3)^2 + \deg(4)^2 \\ &= 4^2 + 1^2 + 1^2 + 1^2 + 1^2 \\ M_1(\Gamma_{\mathbb{Z}_5}) &= 20 \end{aligned}$$

With the same reasoning, we calculate the First Zagreb index for various orders, as showed in the table below.

Table 2.1: The First Zagreb Index

| $n$ | The First Zagreb Index |
|---|---|
| 2 | 2 |
| 3 | 6 |
| 4 | 12 |
| 5 | 20 |
| 7 | 42 |
| 8 | 56 |
| 9 | 72 |

Based on the cases presented in Table 2.1, we discern a pattern in the First Zagreb the coprime graph's index of the integer modulo group modulo n, and this conjecture is confirmed in the theorem below.

**Theorem 2.2.** *The coprime graph of $\mathbb{Z}_n$, where $n$ is a positive integer of the form $n = p^k$ with a prime number $p$ has a formula for the First Zagreb index as follows*

$$M_1(\Gamma_{\mathbb{Z}_n}) = n(n-1)$$

***Proof.*** Suppose that $\Gamma_{\mathbb{Z}_n}$ is the coprime graph of $\mathbb{Z}_n$, where n is a positive integer of the form $n = p^k$ with a prime number $p$. Based on theorem 1, we have a star graph with 0 as the center. □

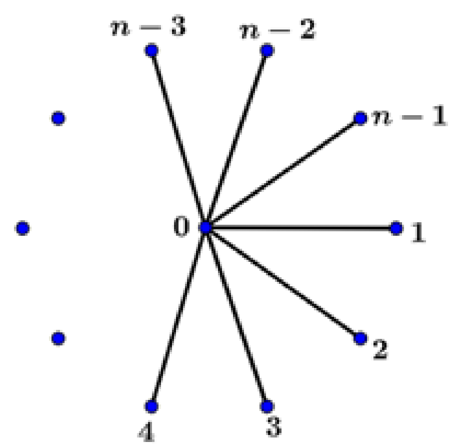


Figure 2.2: Coprime Graph $\mathbb{Z}_5$

we have $\deg(0) = n-1$, and other vertices had degree 1, hence we have

$$\begin{aligned} M_1(\Gamma_{\mathbb{Z}_n}) &= \deg(0)^2 + \deg(1)^2 + \ldots + \deg(n-1)^2 \\ &= (n-1)^2 + 1^2 + 1^2 + \ldots + 1^2 \\ &= (n-1)^2 + (n-1) \\ &= n(n-1). \blacksquare \end{aligned}$$

**Example 2.2.** According to Theorem 2.1, the coprime graph of group $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is a star graph. Therefore, the identity's degree is 4, and a vertex'sother degree is 1. Thus, the following is the Second Zagreb index of $\Gamma_{\mathbb{Z}_5}$.

$$\begin{aligned}
M_2(\Gamma_{\mathbb{Z}_5}) &= \sum_{v_i v_2 \in E(\Gamma_{\mathbb{Z}_5})} \deg(v_i)\deg(v_j) \\
M_1(\Gamma_{\mathbb{Z}_5}) &= \deg(0)\deg(1) + \deg(0)\deg(2) + \deg(0)\deg(3) + \deg(0)\deg(4) \\
M_1(\Gamma_{\mathbb{Z}_5}) &= 4 + 4 + 4 + 4 = 16
\end{aligned}$$

With the same reasoning, we calculate the First Zagreb index for various orders, as showed in the table below.

Table 2.2: The Second Zagreb Index

| $n$ | The Second Zagreb Index |
|---|---|
| 2 | 1 |
| 3 | 4 |
| 4 | 9 |
| 5 | 16 |
| 7 | 36 |
| 8 | 49 |
| 9 | 64 |

Based on the cases presented in Table 2.2, we discern a pattern in the Second Zagreb the coprime graph's index of the integer modulo group modulo n, and this conjecture is confirmed in the theorem below.

**Theorem 2.3.** *The coprime graph of $\mathbb{Z}_n$, where $n$ is a positive integer of the form $n = p^k$ with a prime number $p$, has a formula for the Second Zagreb index as follows*

$$M_2(\Gamma_{\mathbb{Z}_n}) = (n-1)^2$$

***Proof.*** Suppose that $\Gamma_{\mathbb{Z}_n}$ is the coprime graph of $\mathbb{Z}_n$, where $n$ is a positive integer of the form $n = p^k$ with a prime number $p$. Based on Theorem 2.1, we have a star graph with $0$ as the center. Then we have $\deg(0) = n - 1$, and other vertices had degree 1, hence

$$\begin{aligned}
M_2(\Gamma_{\mathbb{Z}_n}) &= \deg(0)\deg(1) + \deg(0)\deg(2) + \ldots + \deg(0)\deg(n-1) \\
&= (n-1)\cdot 1 + (n-1)\cdot 1 + \ldots + (n-1)\cdot 1 \\
&= (n-1)^2. \blacksquare
\end{aligned}$$

$\square$

## 3. Conclusion

The coprime graph for the integer modulo group is essentially a star graph when the order is a prime power. This property, provided by Syarifudin [1], is crucial for determining the graph's topological indices. The formulas for the First Zagreb index and the Second index of a coprime graph for the integer modulo groups are presented in the last two theorems. These numerical topological indices are expected to reveal correlations between elements in the groups based on their order, similar to the original concepts in chemical graph theory. Observing the development of the Index Zagreb theory, there are still the third and fourth Zagreb Index. Therefore, this research can still be further expanded by applying the third and fourth Zagreb Index to the coprime graph of $\mathbb{Z}_n$, where $n = p^k$.

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