J. DIFERENSIAL/e-ISSN:2775-9644
Vol.7(1),2025, 55–72

doi:https://doi.org/10.35508/jd.v7i1.18137

RESEARCH ARTICLE

Numerical Solutions for Linear Integro-Differential Equations Using Shifted Legendre Basis Functions

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Received: 12 August 2024; Revised: 5 November 2024; Accepted: 6 January 2025; Published:15 April 2025.

Abstract:
This research explore the use of Shifted Legendre Basis functions for the numerical solution of a specific class of integro-differential equations. These equations are known for their analytical complexity, making it challenging to derive exact solutions. To address this, we employ an approximate method using Legendre polynomials as basis functions, which provides an efficient approach to finding solutions for these complex problems. The proposed method is computationally efficient, requiring minimal computational resources and storage. The results obtained demonstrate strong agreement with existing solutions found in the literature, validating the accuracy and effectiveness of the approach. This study highlights the potential of Shifted Legendre Basis functions in solving challenging integro-differential equations, offering a reliable alternative to more computationally intensive methods.

Keywords: Shifted Legendre Basis functions, Integro-differential Equations, Numerical Solution, Approximate Method, Computational Efficiency

1. Introduction

Modeling real-world phenomena often leads to the formulation of complex functional equations, including Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs), Integral Equations, and Integro-Differential Equations (IDEs) by [1]. Among these, Integro-Differential Equations uniquely combine both differential and integral terms, making them a powerful tool in various scientific and engineering fields such as physics, biology, economics, and engineering. These equations are integral to understanding and modeling intricate dynamics and behaviors observed in natural and engineered systems [2, 3]. Despite their versatility, Integro-Differential Equations pose significant challenges in both analytical and numerical solutions due to their inherent complexity. Analytical methods, while providing deep insights into the underlying structures of these equations, often become infeasible for real-world problems due to the non-linearity and high-dimensional nature of many systems [4]. This has led to an increased focus on developing efficient numerical methods that can approximate solutions with high accuracy and computational efficiency. To investigate the numerical solution of a specific class of linear Integro-Differential Equations using Shifted Legendre Basis functions. Legendre polynomials, known for their orthogonal properties, are well-suited for constructing approximate solutions to differential equations in [5]. By shifting these basis functions, we aim to enhance their applicability in solving linear IDEs, particularly through collocation techniques [6]. The study emphasizes the potential of this method in providing accurate and computationally efficient solutions, thereby addressing the limitations of traditional analytical approaches. Hence, primary challenge addressed in this study is to evaluate the effectiveness of Legendre polynomial functions, specifically Shifted Legendre Basis functions, in solving a class of linear Integro-Differential Equations [7]. The focus is on implementing and analyzing collocation techniques, which are pivotal in reducing the computational complexity while maintaining high accuracy in the solutions [8]. The aim of this study is to develop and apply a systematic and improved numerical solution for linear Integro-Differential Equations by leveraging modified Legendre Basis functions. To explore and study various methods for solving linear Integro-Differential Equations, with an emphasis on the utility of Legendre Basis functions [9]. The analysis of the effectiveness of both traditional and modified solution methods, particularly focusing on accuracy, stability, and convergence. The accuracy of the numerical solutions against exact or benchmark solutions, thereby validating the proposed approach, to interpret and discuss the results, highlighting the implications of the findings for broader applications in science and engineering [10]. The successful application of Shifted Legendre Basis functions to solve linear Integro-Differential Equations holds significant potential across various scientific and engineering domains. Improved accuracy and reduced computational costs in numerical solutions can lead to a deeper understanding of complex systems, enabling the design of more robust engineering solutions and contributing to advancements in scientific research in [11]. As this study progresses, it is expected to illuminate the broader applicability of these modified basis functions and inspire further research into specialized numerical techniques for solving specific classes of differential and integral equations in [12–14]. By providing a more efficient tool for tackling these challenging equations, this research could pave the way for innovations in fields as diverse as population dynamics, chemical kinetics, and financial modeling [15].

2. Mathematical Formulation

Problem Considered

Legendre polynomial are usually used as polynomial basis function to approximate the solution of integro differential equation such as Linear an non linear fredholm, Voltera and fredholm - Voltera integro differential equation . The general form of the equation are

$$y_n(x) = \sum_{q=0}^{n} p_n(x) y^n(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt \tag{2.1}$$

$$y_n(x) = \sum_{q=0}^{n} p_r(x) y^*(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt \tag{2.2}$$

An unknown function varies with one or more variables, and a differential equation describes the relationship between this function and its rate of change, modeling dynamic systems across various fields. A differential equation has the following generic form

$$F(x, y, y', y'', \ldots, y^(n)) = 0 \tag{2.3}$$

In this case, the unknown function is called y, the independent variable is called x, and the first, second, and nth derivatives of y with respect to x are represented by the variables $y', y'', y''', \ldots,$ and $y(n)$ respectively. These derivatives are related by the function F.

2.1. Integral Equation

An integral equation involves an unknown function under integral signs, modeling phenomena across scientific fields.

The general form of a linear integral equation is given by:

$$u(x) = f(x) + \Lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt = f(x) \tag{2.4}$$

Here, $u(t)$ is the unknown function, $K(x,t)$ is a given kernel function, and $f(x)$ is a known function. Integral equations relate a function to its integral, classified as Fredholm or Volterra, and solved using analytical or numerical methods.

2.2. Linear and Non Linear Differential Equation

Differential equations are categorized as linear or nonlinear based on whether the unknown function and its derivatives appear linearly or nonlinearly within the equation.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{2.5}$$

or other nonlinear combinations. Example of a nonlinear Ordinary differential equations: $dy/dx = ky(1-y)$, where k is a constant. Linear differential equations often allow analytical solutions, while nonlinear equations typically require numerical methods and exhibit more complex system behaviors.

2.3. Homogeneous Differential Equation

Homogeneous Ordinary Differential Equation (ODE):

An ordinary differential equation is considered homogeneous if the sum of any two solutions is also a solution. In the case of a linear ODE, it is typically written in the form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{2.6}$$

Here, y is the unknown function, x is the independent variable, and (x) are coefficients. The term "homogeneous" indicates that the right-hand side is zero. Homogeneous Partial Differential Equation (PDE):

In the context of partial differential equations, a homogeneous equation is one where the sum of any two solutions is also a solution. For example, a linear homogeneous second-order PDE in two variables $u(x,y)$ may look like:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial y} + fu \tag{2.7}$$

a, b, c, d, e, f are constants, and the right-hand side is zero.

3. Legendre Polynomial as a Basis Function

A Synopsis of Legendre Polynomial Basis Functions: A collection of orthogonal polynomials known as legendre polynomial basis functions are essential to many areas of mathematics and physics. These polynomials, which bear the name Adrien-Marie Legendre after the French mathematician, are used in many different domains such as quantum mechanics, signal processing, and numerical analysis. They originate as solutions to Legendre's differential equation.

Orthogonality and Definition

Legendre polynomials, represented by the symbol $P_n(x)$, are orthogonal basis sets that are defined on the interval $[-1, 1]$. One important characteristic of Legendre polynomials that makes numerous mathematical operations and computations simpler is their orthogonality trait. The orthogonality can be stated mathematically as:

$$P_m(x) \cdot P_n(x) \int_{-1}^{1} dx = \frac{2}{2n+1}\delta_{mn} \tag{3.8}$$

where the Kronecker delta is δ_{mn} and is equal to 1. The Legendre polynomials, denoted by $P_n(x)$, are defined on the interval where $\delta_m n$ is the Kronecker delta, which equals 1 when m = n and 0 otherwise. The first few Legendre polynomials are according to Handbook of Mathematical Functions. Dover Publications and Methods of Theoretical Physics. McGraw-Hill Education.
The recurrence relation for Legendre polynomials is given by:

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{1}{2}(3x^2 - 1)\\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3\\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15)\\
P_6(x) &= \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)\\
P_7(x) &= \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x\\
P_8(x) &= \frac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)\\
P_9(x) &= \frac{1}{128}(12155x^9 - 25740x^7 + 18618x^5 - 4620x^3 +)
\end{aligned}$$

Legendre polynomials are essential in solving spherical symmetry problems, especially in physics and engineering, related to spherical harmonics. Legendre's Differential Equation is of the form

$$(1 - x^2 y'' - 2xy' + n(n+1)y = 1) \tag{3.9}$$

and is called Legendre's differential equation where n is a non-negative integer. This equation can also be put in the following form:

$$\frac{y}{dx}(1 - x^2)\frac{dy}{dx} + n(n+1)y = 1 \tag{3.10}$$

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, C_o \neq 0 \tag{3.11}$$

$$y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \tag{3.12}$$

$$y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} \tag{3.13}$$

Putting the above values of y, y' and y" in (2.5), we have

$$\begin{aligned}\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} - x^2y'' &= \sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} \\ &\quad -2x\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} + n(n+1)\sum_{m=0}^{\infty} C_m x^{k+m} \\ &= 0c_o k^{k-1} \\ &= 0 \text{ or } k(k-1)=0, [c_o \neq 0]\end{aligned}$$

which gives two indicial roots $k = k_1 = 1$ and $k = k_2 = 0$. Note that the roots of indicial equation are unequal and differ by an integer. Now, to get the recurrence relation, we equate to zero, the coefficient of x^{k-m+2} in equation (1.11), Thus, we have

$$C_m(k+m)(k+m-1) - C_m - 1(k = m-2+-)(k+m-2+n+1) = 0$$

or

$$C_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} C_m - 2$$

Then, equating to zero , the co efficient of x^{k-1} in (13) , we get

$$C_o(k+1)k = 0$$

it remains valid for k=0: for the explicit polynomial solutions of Legendre Ordinary Differential equation with decreasing power than we have

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n/2}(-1)^k(n,k)()2n-2k,n)x^n - 2k. \tag{3.14}$$

Here are some properties from Legendre polynomials

Lemma I

$P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2) - 1)^n]]$ for $n = 0$, since $0! = 1$, it yields

$$P_0(x) = 1\frac{d}{dx}[(1-x^2)y'] + n(n+1)y = 0 \tag{3.15}$$

for any integer $n > 0$.

$$(x^2-1)\frac{d}{dx}(x^2-1)^n = 2nx(x^2-1)^2 \tag{3.16}$$

By computing the derivatives and moving the product from the right hand side of the product to the left $(n+1)^{th}$ derivatives. It is obtained that

$$\begin{aligned}
\frac{d^n+1}{dx^n+1}[(x^2-1)\frac{d}{dx}(x^2-1)-2nx(x^2-1)^n] &= n(n+1)\frac{d^n}{dx^n}(x^2-1) \\
&\quad +2(n+1)x\frac{d^n+1}{dx^n+1}(x^2-1)^n+(x^2-1) \\
\frac{d^n+2}{dx^n+2}(x^2-1)^n-2n(n+1)\frac{d^n}{dx^n}(x^2+1)-2nx\frac{d^n+1}{dx^n+1} &= -n(n+1)\frac{d^n}{dx^n}(x^2-1)+2x\frac{d(n+1)}{dx(n+1)} \\
&\quad (x^2-1)^2+(x^2-1)\cdot\frac{d(n+1)}{dx(n+2)} \\
&\quad \frac{d}{dx}[(1-x^2)\frac{d}{dx}(\frac{d^n}{dx^n})(x^2-1)] \\
&\quad n(n+1)\frac{d(n)}{dx^n}(x^2-1^n)
\end{aligned}$$

For $P_n(x)$, for example

$$P_n(x)=\sum_{k=0}^{n}$$

Curiously , Legendre polynomials form set of orthogonal contimuous functions over$[-1,1]$

Lemma II

$P_n(x)$of a distinct degrees are orthogonal over [-1,1] , with weighting function $w(x)=1$. It is proved that the orthogonality of two continous functions $f_m(x)$ and $f_n(x)$ over domain $[a,b]$ is equivalent to the invalidity of the inner product $f_m/f_n := \int_a^b f_m(x)f_n(x)w(x)dx$. Taking $w(x)=1$. To show that

$$\int_{-1}^{+1} p_m(x)\cdot p_n(x)dx=0 \quad \forall m\neq n \tag{3.17}$$

Since all $P_n(x)$ satisfy Legendre's ODE we have

$$\begin{aligned}
[(1-x^2)]p_n'(x)]'+n(n+1)P_n &= 0 \\
\text{and } [(1-x^2)]p_m'(x)]'+m(m+1)P_m=0[(1-x^2)]p_n'(x)]'+n(n+1)P_n &= 0 \\
\text{and } [(1-x^2)]p_m'(x)]'+m(m+1)P_m &= 0
\end{aligned}$$

By multiplying and subtracting the above expression will result.

$$(1-x^2)(P_m'P_n'-P_MP_N')]+(m-n)(m+n+1)P_mP_n=0$$

Mainly , the integration of both sides over the domain $|-1,1]$

$$\int_{-1}^{+1}\frac{d}{dx}[(1-x^2)(P_m'P_n-P_mP_n')]dx+(m-n+1)\int_{-1}^{+1}p_mp_ndx=0$$

and this always as $(m-n)(m+n+1)\int_{-1}^{+1}p_mp_ndx=0\,\forall$ non negatives values of $m\neq n$. Where $P_m(x)$ and $P_n(x)$ are said to be orthogonal in the interval $0\leq x\leq 1$.

3.1. Shifted Legendre Polynomial

Shifted Legendre polynomials, represented by $P_n^{(\alpha)}(x)$, are an extension of classical Legendre polynomials incorporating a shift parameter α. They are orthogonal over non-standard intervals, vital for diverse mathematical modeling applications.

$$L*_k+1(x) = \frac{(2k+1)(2x-1)}{k+1} L*_k(x) - \frac{k}{(k+1)} L*(k-1)(x) \tag{3.18}$$

where

$$\begin{gathered}
L*_0(x) = 1 \\
L*_1(x) = 2x - 1 \\
L*_2(x) = 6x^2 - 6x + 1 \\
L*_3(x) = 20x^3 - 30x^2 + 12x + 1 \\
L*_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1 \\
L*_5(x) = 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1 \\
L*_6(x) = 924x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1 \\
L*_7(x) = 3432x^7 - 12012x^6 + 1663x^5 - 11550x^4 + 4200x^3 - 756x^2 + 56x - 1 \\
L*_8(x) = 12870x^8 - 51480x^7 + 84084x^6 - 72072x^5 + 34650x^4 - 9240x^3 + 1260x^2 - 72x + 1
\end{gathered}$$

4. Collocation algorithm

Here, standard Collocation Method is used for solving one Dimensional Ordinary Integro Differential Equation using the Shifted Legendre Polynomial are basis function Consider the ordinary Integro Differential Equation of the general form

$$U^m(x) + f(x)U(x) + \lambda \int a-^b(x)w(x,t)U(t)dt = g(t) \tag{4.19}$$

and the assumed solution of the form

$$U_N(x) = \sum_{i=0}^{N} a_i \phi_i(x) \tag{4.20}$$

For the purpose of discussion the assumed approximate solution is of the form

$$U^{\Lambda}(x) = \sum_{i=0}^{N} a_i \phi_i(x) + H_N(x) \tag{4.21}$$

where a_i are constant to be determined, $\phi(x)$ are Legendre polynomial, $H_N(x)$ are the perturbed terms. Substitute equation(3.3) into equation (3.1) to obtain as follows

$$\tilde{U}_N^m(x) + f(x)\tilde{U}_N(x) + \lambda \int ab(x)w(x,t)U_N(t)dt = g(x) + H_N(x) \tag{4.22}$$

where $\tilde{U}_N$ is n^{th} derivative of $U_N(x_0)$.

$$\left[\sum_{i=0}^{N} a_i\phi_i(x) + H_N(x)\right]^n + f(x)\left[\sum_{i=0}^{N} a_i\phi_i(x) + H_N(x)\right] + \lambda\int_a^b w(x,t)\left[\sum_{i=0}^{N} a_i\phi_i(t)\right]dt = g(x)$$

$$\sum_{k=0}^{N} a_i\phi_i(x) + H_N(x) = \bar{U}(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \ldots + a_N\phi_N(x) + H_N(x)$$
$$\bar{u}'(x) = a_0\phi_0'(x) + a_1\phi_1'(x) + a_2\phi_2'(x) + a_3\phi_3'(x)...a_N\phi_N'(x) + H_N'(x)$$
$$\bar{u}''(x) = a_0\phi_0''(x) + a_1\phi_1''(x)a_2\phi_2''(x) + a_3\phi_3''(x)...a_N\phi_N''(x) + H_N''(x)$$
$$\vdots$$
$$\bar{U}_N^m(t) = a_o\phi_0^n(x) + a_1\phi_1^n(x) + a_2\phi_2^n(x) + a_3\phi_3^n(x) + ... + a_N\phi_N^m(x) + H_N^n(x)$$
$$\bar{U}(t) = a_0\phi_0(t) + a_1\phi_1(t) + a_2\phi_2(t) + a_3\phi_3(t) + \ldots + a_N\phi_N(t) + H_N(t)$$

substitute 3.6, 3.7,and 3.8 into equation 3.4 to obtain as follows;

$$\begin{aligned} a_0\phi_0^m(x) + a_1\phi_1^m(x) + a_2\phi_2(x)^m \quad &+...+ \quad a_n\phi_N^m(x) + H_N(x) + f(x_0)[a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) \\ &+...+ \quad a_N\phi_N(x_0 + H_N(x))] + \lambda\int_a^{b(x)} w(x,t)[a_0\phi_0(t) + a_1\phi_1(x) \\ &+ \quad a_2\phi_2(t) + ... + a_N\phi_N(x_0) + H_N(x)]dt \\ &= \quad g(x) + H_N(x) \end{aligned} \tag{4.23}$$

Further simplification of equation 3.11 gives

$$\left[\phi_0^m(x) + \phi_0(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_0(t)dt\right]a_0 + \left[\phi_1^m(x) + \phi_1(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_1(t)dt\right]a_1 +$$
$$\left[\phi_2^m(x) + \phi_2(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_2(t)dt\right]a_2 + \ldots + \left[\phi_N^m(x) + \phi_N^m f(x) + \lambda\int_a^b(x)w(x,t)\phi_N^M(t)dt\right]a_N$$
$$+H_N^m(X) + f(x)H_N(x) + \lambda\int_a^b(x)w(x,t)H_N(t)dt = g(x) + H_N(x) \tag{4.24}$$

$$\left[\phi_0^m(x) + \phi_0(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_0(t)dt\right]a_0 + \left[\phi_1^m(x) + \phi_1(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_1(t)dt\right]a_1 +$$
$$\left[\phi_2^m(x) + \phi_2(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_2(t)dt\right]a_2 + \ldots + \left[\phi_N^m(x) + \phi_N(x)f(x) + \lambda\int_a^b(x)w(x,t)\phi_N(t)dt\right]a_N$$
$$+H_N(x)^m(x) + f(x)H_N(x) + \lambda\int_a^b(x)w(x,t)H_N(t)dt = g(x) + H_N(x)$$
$$H_N^m(x) = \sum_{p=1}^{m} \tau_p L_{N-P+1} \tag{4.25}$$

n represent order of Integro-Differential Equation.

$$\begin{aligned} H_N(x) &= \tau_1 L_N(x) + \tau_2 L_{N-1}(x) + \tau_3 L_{N-2}(x) + \ldots + \tau_n L_{N-m+1}(x) \\ H_N'(x) &= \tau_1 L_N'(x) + \tau_2 L_N' - 1(x) + \tau_3 L_{N-2}'(x) + ...\tau_n L_{N-m+1}'(x) \\ H_N^m(x) &= \tau_1 L_N^m(x) + \tau_2 L_{N-1}^m(x) + \tau_3 L_{N-2}^m(x) + \ldots + \tau_n L_{N-m+1}^m(x) \\ H_N(t) &= \tau_1 L_N(t) + \tau_2 L_{N-1} + \tau_3 L_{N-2}(t) + \ldots + \tau_n L_{N-m+1} \end{aligned} \tag{4.26}$$

substitute equation, 3.14, 3.15, and 3.17 into equation 3.13 to obtain equation 3.18

$$\left[\phi_0^n(x) + \phi_0(x)f(x) + \lambda \int_a^b (x)w(x,t)\phi_0(t)dt\right]a_0 + \left[\phi_1^n(x) + \phi_1(x)f(x) + \lambda \int_a^b (x)w(x,t)\phi_1(t)dt\right]a_1 +$$
$$\left[\phi_2^n(x) + \phi_2(x)f(x) + \lambda \int_a^b (x)w(x,t)\phi_2(t)dt\right]a_2 + \ldots + \left[\phi_N^m(x) + \phi_N(x)f(x) + \lambda \int_a^b (x)w(x,t)\phi_N(t)dt\right]a_N$$
$$+\tau_1 L_N^n(x) + \tau_2 L_{N-1}^n(x) + \ldots + \tau L_{N-m+1}^n + f(x)\left[\tau_1 L_N(x) + \tau_2 L_{N-1}(x) + \tau_n L_{N-m+1}(x)\right] + \lambda \int_a^b (x)w(x,t)$$
$$\left[\tau_1 L_N(t) + \tau_2 L_{N-1}(t) + \ldots + \tau_n L_{N-m+1}(x)\right]dt = g(x) + \tau_1 L_N(x) + \tau_2 L_{N-1}(x) + \ldots + \tau_n L_{N-m+1} \quad (4.27)$$

Collocating equation (3.18)

$$x_i = a + \frac{a + (b-a)i}{N} \quad (4.28)$$

at point and N is the degree of the approximant used. Hence, it gives rise to $[N+1]$ algebraic linear system of equation are then solved using elimination methods to obtain the unknown constants at $i > 0$ which are then substituted into equation (3) to obtain the appropriate solution

4.1. Discussion of Results

This method analyses the use of Shifted Legendre Basis Functions in numerically solving linear integro-differential equations. It covers aspects like numerical accuracy, flexibility, comparability with alternative techniques, stability, computational efficiency, usefulness, and implications for further study. The work examines numerical correctness and convergence behavior by comparing calculated results with known analytical solutions and fine-tuning the spatial grid. Findings show that Shifted Legendre Basis Functions are effective in solving linear integro-differential equations

4.2. Example 1

Consider the second order linear fredholm integro-differential equation (Daraina)

$$y''(x) = e^x - x + \int_0^1 xty(t)dt \quad y(0) = 1, y'(0) = 1$$

with exact solution is $y(x) = e^x$

Solution:

Let $U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x)$ and the perturbed assumed solution be

$$U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x) + H_N(x)$$

where

$$H_N(x) = \sum_{p=1}^{m} \tau_p L_{N-P+1}$$

$$\sum_{k=0}^{N} a_i\phi_i(x) + H_N(x) = \bar{U}(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \ldots + a_N\phi_N(x) + H_N(x)$$
$$\bar{u}'(x) = a_0\phi_0'(x) + a_1\phi_1'(x) + a_2\phi_2'(x) + a_3\phi_3'(x)...a_N\phi_N'(x) + H_N'(x)$$
$$\bar{u}''(x) = a_0\phi_0''(x) + a_1\phi_1''(x)a_2\phi_2''(x) + a_3\phi_3''(x)...a_N\phi_N''(x) + H_N''(x)$$
$$\vdots$$
$$\bar{U}_N^m(t) = a_o\phi_0^n(x) + a_1\phi_1^n(x) + a_2\phi_2^n(x) + a_3\phi_3^n(x) + ... + a_N\phi_N^m(x) + H_N^n(x)$$
$$\bar{U}(t) = a_0\phi_0(t) + a_1\phi_1(t) + a_2\phi_2(t) + a_3\phi_3(t) + \ldots + a_N\phi_N(t) + H_N(t)$$

Consider for case $N = 5$

$$y(x) = y_4(x) = \sum_{r=0}^{4} a_r L_r(x) = a_0 + a_1(2x-1) + a_2(6x^2-6x+1) + a_3(20x^3-30x^2+12x+1)$$
$$+a_4(70x^4-140x^3+90x^2-20x+1) + a_5(252x^5-630x^4+560x^3-210x^2+30x-1)$$

Substitute equations 4.3, 4.4, 4.5, into equation 4.1 we obtain as follows:

$$2a_1 + (12x-6)a_2 + (60x^2-60x+12)a_3 + (280x^3-420x^2+180x-20)a_4 + (1260x^4-2520x^3$$
$$+1680x^2+420x+30)a_5 + (1260x^4-2520x^3+1680x^2+420x+30)\tau_1 - x\Bigg(a_0\frac{t^2}{2} + a_1(\frac{2t^3}{3} - \frac{t^2}{2})$$
$$+a_2(\frac{6t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}) + a_3(\frac{20t^5}{5} - \frac{30t^4}{4} + \frac{12t^3}{3} - \frac{t^2}{2}) + a_4(\frac{70t^6}{6}\frac{140t^5}{5} + \frac{90t^4}{4} - \frac{20t^3}{3} + \frac{t^2}{2})$$
$$+a_5(\frac{252t^7}{7} - \frac{630t^6}{6} + \frac{560t^5}{5} - \frac{210t^4}{4} + \frac{30t^3}{3} - \frac{t^2}{2})\tau_1]\Bigg) = e^{x-x}$$

Now applying the initial condition $U_5(0)$ = 1 to get

$$a_1 + a_2 + a_3 + a_4 + a_5 = 1$$

Differentiating equation 4.10 and applying the initial condition $U_5' = -3$ to get

$$2a_1 - 6a_2 + 12a_3 - 20a_4 = 1$$

This equation (4.9) is collocating at point $x = x_i$, where $x_i = -1 + \frac{2i}{4}; i = 1, 2, 3, 4, 5.$ and other simplification gives

$$-0.1068769702e^{-1}. - 0.3018677642e^{-2}\tau_2 - 0.5218784e^{-3}\tau_1 + 0.3208768098e^{-2}a_4$$
$$-0.534566876293e^{-2}a_3 + 0.7481733744e^{-2}a_2 - 0.9215030710e^{-2}a_1 + 0.1018571084e^{-1}a_0 = 0$$

Solving simultaneously and substituting the values of the constants into the trial solution gives: $U_5 = 1.0000076 + 0.999709x + 0.50477x^2 + 0.1539x^3 + 0.7012e^{-1}x^4$
The numerical result obtained from example 2 is depicted below

Table 4.1: Comparison of numerical solution and Exact solution with Daraina et al

(x,t)	Exact Solution	Approximate	Absolute Error	Daraina et al.(2016)
0.0	1.0000000000	1.0000098000	$9.8000e^{-06}$	$0.00000000e^{+0}$
0.1	1.1051709180	1.1051662120	$4.7060e^{-06}$	$1.66666667e^{-03}$
0.2	1.2214027580	1.2214015920	$1.1660e^{-06}$	$6.09388620e^{-03}$
0.3	1.3498588080	1.3498557720	$3.0360e^{-06}$	$1.32017875e^{-02}$
0.4	1.4918246980	1.4918128720	$1.1826e^{-05}$	$2.29140636e^{02}$
0.5	1.6487212710	1.6487013000	$1.9971e^{-05}$	$3.51578404e^{-02}$
0.6	1.8221188000	1.8220937520	$2.5048e^{-05}$	$6.69648304e^{-02}$
0.8	2.0137527070	2.0137072120	$4.5495e^{-05}$	$8.63983845e^{-02}$
0.9	2.2255409280	2.2254029520	$1.3798e^{-04}$	$1.08103910e^{-01}$
1.0	2.4596031110	2.4591865320	$4.1658e^{-04}$	$1.32023989e^{-01}$

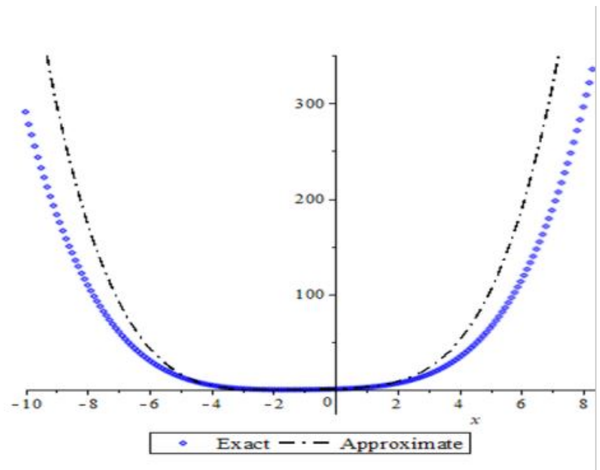

Figure 4.1: Comparison between result of exact and approximate solution for example 1

4.3. Example 2

Consider the third order integro differential equation

$$y'''(x) = sinx - x - \int_0^1 xty'(t)dt \quad y(0) = 1, y(1) = 0, y'(0) = -1$$

with the exact solution $y'(x) = cosx$

Solution:

Let $U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x)$ and the perturbed assumed solution be

$$U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x) + H_N(x)$$

where $H_N(x) = \sum_{p=1}^{m} \tau_p L_{N-P+1}$

$$\sum_{k=0}^{N} a_i\phi_i(x) + H_N(x) = \bar{U}(x) = a_0\phi_0(x) + a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x) + \ldots + a_N\phi_N(x) + H_N(x)$$
$$\bar{u'}(x) = a_0\phi_0'(x) + a_1\phi_1'(x) + a_2\phi_2'(x) + a_3\phi_3'(x) ... a_N\phi_N'(x) + H_N'(x)$$
$$\bar{u''}(x) = a_0\phi_0''(x) + a_1\phi_1''(x) a_2\phi_2''(x) + a_3\phi_3''(x) ... a_N\phi_N''(x) + H_N''(x)$$
$$\vdots$$
$$\bar{U}_N^m(t) = a_o\phi_0^n(x) + a_1\phi_1^n(x) + a_2\phi_2^n(x) + a_3\phi_3^n(x) + ... + a_N\phi_N^m(x) + H_N^n(x)$$
$$\bar{U}(t) = a_0\phi_0(t) + a_1\phi_1(t) + a_2\phi_2(t) + a_3\phi_3(t) + \ldots + a_N\phi_N(t) + H_N(t)$$

Consider for case $N = 5$

$$y(x) = y_4(x) = \sum_r -0^4 a_r L_r(x) = a_0 + a_1(2x-1) + a_2(6x^2 - 6x + 1) + a_3(20x^3 - 30x^2 + 12x + 1)$$
$$+ a_4(70x^4 - 140x^3 + 90x^2 - 20x + 1) + a_5(252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1)$$

substitute equations 4.3, 4.4, 4.5, into equation 4.1 we obtain as follows :

$$2a_1 + (12x-6)a_2 + (60x^2 - 60x + 12)a_3 + (280x^3 - 420x^2 + 180x - 20)a_4 + (1260x^4 - 2520x^3$$
$$+1680x^2 + 420x + 30)a_5 + (1260x^4 - 2520x^3 + 1680x^2 + 420x + 30)\tau_1$$
$$-x\int_0^1 \Bigg[t(a_0 + a_1(2t-1) + a_2(6t^2 - 6t + 1) + a_3(20t^3 - 30t^2 + 12t - 1)$$
$$+a_4(70x^4 - 140x^3 + 90x^2 - 20x + 1) + \tau_1]dt = e^{x-x}\Bigg]$$

$$2a_1 + (12x-6)a_2 + (60x^2 - 60x + 12)a_3 + (280x^3 - 420x^2 + 180x - 20)a_4 + (1260x^4 - 2520x^3$$
$$+1680x^2 + 420x + 30)a_5 + (1260x^4 - 2520x^3 + 1680x^2 + 420x + 30)\tau_1 - x\Bigg[a_0\frac{t^2}{2} + a_1(\frac{2t^3}{3} - \frac{t^2}{2})$$
$$+a_2(\frac{6t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}) + a_3(\frac{20t^5}{5} - \frac{30t^4}{4} + \frac{12t^3}{3} - \frac{t^2}{2}) + a_4(\frac{70t^6}{6}\frac{140t^5}{5} + \frac{90t^4}{4} - \frac{20t^3}{3} + \frac{t^2}{2})$$
$$+a_5(\frac{252t^7}{7} - \frac{630t^6}{6} + \frac{560t^5}{5} - \frac{210t^4}{4} + \frac{30t^3}{3} - \frac{t^2}{2})\tau_1\Bigg] = sinx - x\Bigg]$$

Now applying the initial condition $U_5(0) = 1$ to get

$$a_1 + a_2 + a_3 + a_4 + a_5 = 1$$

Differentiating equation 4.10 and applying the initial condition $U_5' = -3$ to get

$$2a_1 - 6a_2 + 12a_3 - 20a_4 = 1$$

This equation (4.9) is collocating at point $x = x_i$, where $x_i = -1 + \frac{2i}{4}; i = 1, 2, 3, 4, 5.$ and other simplification gives

$$-0.1068889702e^{-1}. - 0.3018967642e^{-2}\tau_2 - 0.52160184e^{-3}\tau_1 + 0.3203492098e^{-2}a_4$$
$$-0.5342491293e^{-2}a_3 + 0.7481733744e^{-2}a_2 - 0.9215030710e^{-2}a_1 + 0.1018571084e^{-1}a_0 = 0$$

solving simultaneously and substituting the values of the constants into the trial solution gives:

$$U_5 = 1.0000098 + 0.999608x + 0.50357x^2 + 0.1539x^3 + 0.6012e^{-1}x^4$$

The numerical result obtained from example 2 is depicted below

Table 4.2: Comparison of numerical solution and Exact solution with Daraina et al

(x,t)	Exact Solution	Approximate	Absolute Error	Daraina et al.(2016)
0.0	1.00000000	1.00000620	$6.2000e^{-06}$	$0.00000000e^{+00}$
0.1	1.21709180	1.217066212	$2.5588e^{-08}$	$1.00118319e^{-02}$
0.2	1.43175800	1.43159200	$1.1660e^{-06}$	$2.78651355e^{-02}$
0.3	1.39880000	1.39872000	$8.0000e^{-05}$	$5.08730892e^{-02}$
0.4	1.49182469	1.49181287	$1.1826e^{-07}$	$7.55356316e^{-02}$
0.5	1.64872127	1.64870130	$1.9971e^{-05}$	$9.71888592e^{-02}$
0.6	1.82211880	1.82209370	$2.5100e^{-07}$	$1.09551714e^{-01}$
0.7	2.13775277	2.13707212	$6.8065e^{-08}$	$1.04133232e^{-01}$
0.8	2.22554090	2.22540295	$1.3795e^{-08}$	$1.94512700e^{-02}$
0.9	2.46671100	2.46673200	$2.1000e^{-06}$	$1.00034260e^{-02}$
1.0	2.77238000	2.77230000	$1.0000e^{-05}$	$1.55147712e^{-01}$

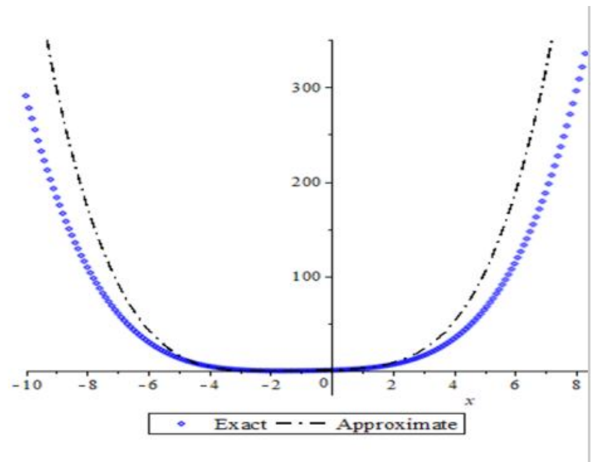

Figure 4.2: Comparison between result of exact and approximate solution for example 2

4.4. Example 3

Consider the second order integro-differential equation

$$u^2 = 1 + x + \frac{1}{6}x^3 + \int tu(t), \quad t = 0y(0) = 0, y(1) = 1$$

$y(exact) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} + \frac{x^4}{4}$ Consider the second order linear fredholm integro-differential equation (Daraina)

$$y''(x) = e^x - x + \int_0^1 xty(t)dt \quad y(0) = 1, y'(0) = 1$$

with exact solution is $y(x) = e^x$

Solution:

Let $U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x)$ and the perturbed assumed solution be

$$U_N(x) = \sum_{r=0}^{N} a_i\phi_i(x) + H_N(x) \tag{4.29}$$

where $H_N(x)=\sum_{p=1}^{m}\tau_p L_{N-P+1}$

$$\sum_{k=0}^{N}a_i\phi_i(x)+H_N(x)=\bar{U}(x)=a_0\phi_0(x)+a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+\ldots+a_N\phi_N(x)+H_N(x)$$
$$\bar{u}'(x)=a_0\phi_0'(x)+a_1\phi_1'(x)+a_2\phi_2'(x)+a_3\phi_3'(x)...a_N\phi_N'(x)+H_N'(x)$$
$$\bar{u}''(x)=a_0\phi_0''(x)+a_1\phi_1''(x)a_2\phi_2''(x)+a_3\phi_3''(x)...a_N\phi_N''(x)+H_N''(x)$$
$$\vdots$$
$$\bar{U}_N^m(t)=a_o\phi_0^n(x)+a_1\phi_1^n(x)+a_2\phi_2^n(x)+a_3\phi_3^n(x)+...+a_N\phi_N^m(x)+H_N^n(x)$$
$$\bar{U}(t)=a_0\phi_0(t)+a_1\phi_1(t)+a_2\phi_2(t)+a_3\phi_3(t)+\ldots+a_N\phi_N(t)+H_N(t)$$

Consider for case $N=5$

$$\begin{aligned}y(x)=y_4(x) &= \sum_r -0^4 a_r L_r(x)=a_0+a_1(2x-1)+a_2(6x^2-6x+1)+a_3(20x^3-30x^2+12x+1)\\ &\quad +a_4(70x^4-140x^3+90x^2-20x+1)+a_5(252x^5-630x^4+560x^3-210x^2+30x-1)\end{aligned}$$

substitute equations 4.3, 4.4, 4.5, into equation 4.1 we obtain as follows :

$$\begin{aligned}&2a_1+(12x-6)a_2+(60x^2-60x+12)a_3+(280x^3-420x^2+180x-20)a_4+(1260x^4-2520x^3\\ &+1680x^2+420x+30)a_5+(1260x^4-2520x^3+1680x^2+420x+30)\tau_1-x\Bigg(a_0\frac{t^2}{2}+a_1(\frac{2t^3}{3}-\frac{t^2}{2})\\ &+a_2(\frac{6t^4}{4}-\frac{t^3}{3}+\frac{t^2}{2})+a_3(\frac{20t^5}{5}-\frac{30t^4}{4}+\frac{12t^3}{3}-\frac{t^2}{2})+a_4(\frac{70t^6}{6}\frac{140t^5}{5}+\frac{90t^4}{4}-\frac{20t^3}{3}+\frac{t^2}{2})\\ &+a_5(\frac{252t^7}{7}-\frac{630t^6}{6}+\frac{560t^5}{5}-\frac{210t^4}{4}+\frac{30t^3}{3}-\frac{t^2}{2})\tau_1\Bigg)=e^{x-x}\end{aligned}$$

Now applying the initial condition $U_5(0)$ = 1 to get

$$a_1+a_2+a_3+a_4+a_5=1$$

Differentiating equation 4.10 and applying the initial condition $U_5'=-3$ to get

$$2a_1-6a_2+12a_3-20a_4=1$$

This equation (4.9) is collocating at point $x=x_i$, where $x_i=-1+\frac{2i}{4}; i=1,2,3,4,5.$ and other simplification gives

$$\begin{aligned}&-0.1068889702e^{-1}-0.3018967642e^{-2}\tau_2-0.52160184e^{-3}\tau_1+0.3203492098e^{-2}a_4\\ &-0.5342491293e^{-2}a_3+0.7481733744e^{-2}a_2-0.9215030710e^{-2}a_1+0.1018571084e^{-1}a_0=0\end{aligned}$$

solving simultaneously and substituting the values of the constants into the trial solution gives:

$$U_5=1.0000098+0.999608x+0.50357x^2+0.1539x^3+0.6012e^{-1}x^4$$

As obtained the results for Example 3 iterations is depicted below

Table 4.3: Comparison of numerical solution and Exact solution with Daraina et al

(x,t)	Exact Solution	Approximate	Absolute Error	Daraina et al.(2016)
0.0	1.0000000000	1.0000160000	$1.6000e^{-05}$	$0.0000e^{0}$
0.1	1.1051708340	1.1053313500	$1.6052e^{-04}$	$1.5900e^{02}$
0.2	1.2214000000	1.2227436000	1.3436e-03	$9.500e^{02}$
0.3	1.3498375000	1.3543887500	$4.5512e^{-03}$	$1.9700e^{1}$
0.4	1.4917333340	1.5025840000	$1.0851e^{-02}$	$1.4900e^{1}$
0.5	1.6484375000	1.6698277500	$2.1390e^{-02}$	$1.4900e^{1}$
0.6	1.8214000000	1.8587996000	$3.7400e^{-02}$	$1.9703e^{1}$
0.7	2.0121708340	2.0723603500	$6.0190e^{-02}$	$9.5007e^{2}$
0.8	2.2224000000	2.3135520000	$9.1152e^{-02}$	$3.5908e^{1}$
0.9	2.4538375000	2.5855977500	$1.3176e^{-01}$	$1.5987e^{1}$
1.0	2.7083333340	2.8919020000	$1.8357e^{-01}$	$1.0007e^{-1}$

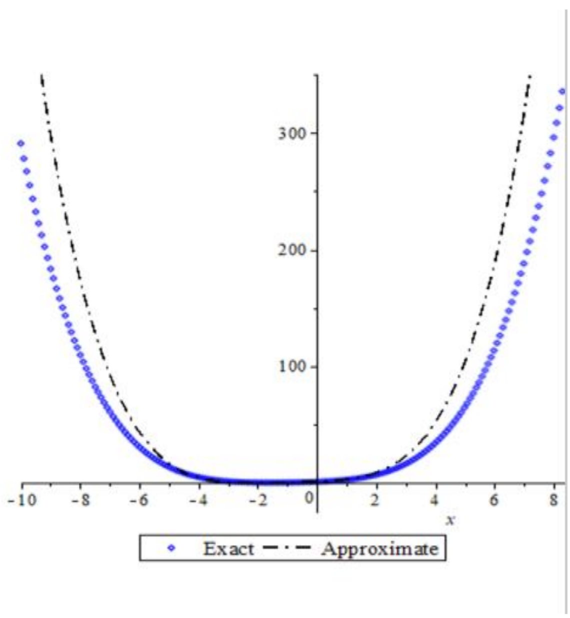

Figure 4.3: Comparison between result of exact and approximate solution for example 3

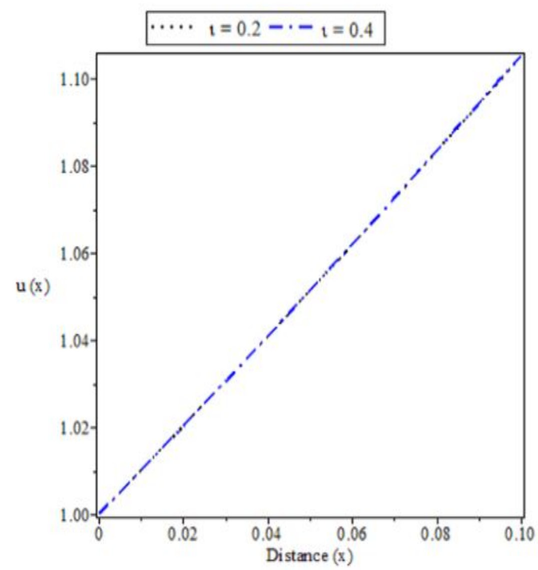

Figure 4.4: Comparison between result of exact and approximate solution for example 3

From the tabular representation of results, it is observed that the following observations were considered as factors in variation between numerical solution and exact solution

1. Truncation and Discretization Errors: In numerical methods, continuous functions are represented by discrete values at specific points. Using basis functions like the shifted Legendre polynomials involves truncating the series expansion to a finite number of terms. This truncation introduces a truncation error, as only a finite number of terms approximate the solution. Moreover, discretizing the IDE can also lead to discretization errors due to the finite representation of continuous operators.
2. Approximation of Basis Functions: Shifted Legendre polynomials, although effective in many cases, are still approximations when used in a truncated series. The accuracy of the numerical solution depends heavily on the degree of the polynomial expansion and the suitability of the Legendre basis functions for the problem's specific boundary and initial conditions. For com-

plex integro-differential equations, the chosen polynomial order might not capture all aspects of the solution's behavior, leading to approximation errors.

3. Integration and Differentiation Approximations: Integro-differential equations involve both integral and differential terms, which may need to be approximated separately. In numerical solutions, integrals are often approximated by quadrature rules, and differentiation is approximated by finite differences or polynomial-based derivatives. The inherent approximation in these methods can introduce errors, especially for higher-order derivatives or integrals over large intervals.
4. Boundary Condition Handling: Numerical methods require enforcing boundary or initial conditions in the context of basis functions. These conditions might not align precisely with the polynomial basis functions, leading to minor adjustments or approximations that add to the overall error. This effect is especially notable in boundary layers where solutions may have steep gradients, as polynomial basis functions struggle to capture such localized behaviors accurately.
5. Computational Precision: Numerical computations are limited by machine precision. Small round-off errors in arithmetic operations accumulate, particularly in iterative or multi-step computations. While these errors are typically small, they can become significant in high-dimensional problems or when the method involves large matrices that amplify round-off errors.
6. Algorithm Convergence: Some algorithms used for numerical solutions of IDEs may converge slowly or not at all, depending on the initial guess, matrix conditioning, or problem-specific parameters. Poor convergence behavior can increase errors, as a solution that is not well-converged will inherently differ from the exact solution.

5. Conclusion

Integro-differential equations, which involve derivatives and integrals of unknown functions, can be effectively addressed using Legendre polynomials—orthogonal polynomials defined on the interval $[-1, 1]$. These polynomials allow for the representation of the unknown function as a series expansion, simplifying the process of manipulation and analysis. The orthogonality of Legendre polynomials significantly reduces computational complexity, enabling efficient evaluation of integrals and facilitating the derivation of closed-form solutions. In applying Shifted Legendre Basis Functions to the numerical solution of linear integro differential equations, this method demonstrates its effectiveness, versatility, and practical utility. The results offer valuable insights into efficient approximation of solutions. The use of shifted Legendre basis functions offers an effective way to approximate solutions to linear integro-differential equations with reasonable accuracy, even when dealing with complex boundary conditions and also flexibility in handling complex equations. This numerical approach allows for the treatment of both integral and differential components within integro-differential equations, making it a versatile option for a wide range of Integro-Differential Equations.

6. Recommendation

In order to employ shifted Legendre basis functions to solve linear integro-differential equations, we must first analyze Optimize basis function selection, future work could explore adaptive or problem-specific basis functions that better capture local solution behaviors, especially in regions with steep gradients or boundary layers and increase computational precision with efficiency as this will bring about implementing higher-precision arithmetic and improving convergence algorithms could further reduce computational errors, particularly for high-dimensional or stiff integro-differential problems.

Financial support

No funding was received during the course of this research.

Acknowledgement

Authors appreciate enormous effort of respective reviewers that will improve this manuscript.

Declaration of interest

The authors declare that there is no conflict of interest.

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Citation IEEE Format:

O. O. Babalola, A. F. Adebisi, and K. A. Odeyemi. *"Numerical Solutions for Linear Integro-Differential Equations Using Shifted Legendre Basis Functions"*, *Jurnal Diferensial*, vol. 7(1), pp. 55-72,2025.