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RESEARCH ARTICLE

Some Conditions for the Boundedness of Commutators of Fractional Integrals on Generalized Weighted Morrey Spaces

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Abstract:
In this paper, we investigate the boundedness of commutators generated by $b \in BMO$ and fractional integrals I_α from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$ for $1 < p < q < \infty$. We obtain some new conditions for the pair (ψ_1, ψ_2) of the functions ψ_1 and ψ_2 on $\mathbb{R}^n \times (0, \infty)$ that ensure the boundedness properties.

Keywords: Commutator, Fractional Integral, Generalized Weighted Morrey Space

1. Introduction

Morrey spaces were first introduced by C. B. Morrey in 1938 [1]. Many authors then studied commutators of a certain operator on the spaces [2], [3]. Generalized Morrey spaces were then introduced as in [4]. In 2009, Komori and Shirai [5] introduced the weighted Morrey space $L^{p,\kappa}(w)$. The two spaces $L^{p,\varphi}$ and $L^{p,\kappa}(w)$ were then generalized by Guliyev as generalized weighted Morrey space $L^{p,\varphi}(w)$.

The topic of boundedness operator on function spaces was investigated by many mathematicians particularly related to the Morrey spaces. Fractional integrals and Morrey spaces have many applications in harmonic analysis and PDEs. The application of the operator as well as the function spaces may be found in [6].

The boundedness of fractional integral I_α on generalized weighted Morrey spaces is well-known, see [7, 8] for example. The boundedness of commutators of fractional integrals on weighted Lebesgue space was studied by Segovia and Torrea [9]. Shirai [10] gave the necessary and sufficient condition for the boundedness of commutators of fractional integrals on classical Morrey space under some assumptions. Guliyev [11] gave the conditions on the parameter function ψ_1 and ψ_2 that ensure the boundedness of commutators. The conditions involved the logaritmic natural function. In this paper, we give some new conditions for the pair ψ_1 and ψ_2 that ensure the boundedness of commutator of fractional integrals from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$. In precise, the followings are our main results.

Theorem 1.1. *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, $w \in A_{p,q}$ and I_α be the fractional integrals. Suppose that ψ_1, ψ_2 be positive functions on $\mathbb{R}^n$ such that there exists $C > 0$ and $\beta > 0$ for which*

$$\sup_{(a,r)\in\mathbb{R}^n\times(0,\infty),\lambda\geq 2} \frac{\lambda^\beta}{\psi_2(a,r)} \int_{\lambda r}^{\infty} \psi_1(a,s)\frac{ds}{s} \leq C.$$

If $b \in BMO$, then $[b, I_\alpha]$ is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$. Precisely, there is a constant $D > 0$ such that

$$\|[b, I_\alpha]f\|_{\mathcal{M}^{\psi_2}_{q,w^q}} \leq D\|b\|_*\|f\|_{\mathcal{M}^{\psi_1}_{q,w^q}}, \quad f \in \mathcal{M}^{p,w^p}_{\psi_1}.$$

Corollary 1.1. *Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$, and I_α be Riesz potential. Assume that there is a constant $\beta > 0$ such that*

$$\sup_{r<s<\infty} \psi_1(a, s)\frac{w^p(B(a, s))^{\frac{1}{p}}}{w^q(B(a, s))^{\frac{1}{q}}} s^\beta \leq C\psi_2(a, r)r^\beta$$

for all $(a, r) \in \mathbb{R}^n \times (0, \infty)$. If $b \in BMO$ and $w \in A_{p,q}$, then $[b, I_\alpha]$ is bounded from $\mathcal{M}^{p,w^p}_{\psi_1}$ to $\mathcal{M}^{q,w^q}_{\psi_2}$. Precisely, there is a constant $D > 0$ such that

$$\|[b, I_\alpha]f\|_{\mathcal{M}^{\psi_2}_{q,w^q}} \leq D\|b\|_*\|f\|_{\mathcal{M}^{\psi_1}_{q,w^q}}, \quad f \in \mathcal{M}^{p,w^p}_{\psi_1}.$$

2. Some Definitions and Previous Results

Let $a \in \mathbb{R}^n$ and $r > 0$. We denote $B(a, r)$ as an open ball centered at a with radius r. For the Ball $B = B(a, r)$ and $k > 0$, kB denotes $B(a, kr)$, namely the ball with the same center as B but with radius k times r. Moreover, $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^n$. A weight w is a nonnegative locally integrable functions on $\mathbb{R}^n$ taking values in the interval $(0, \infty)$ almost everywhere [5].

For the weight w, $1 \leq p < \infty$, and E a measurable subset of $\mathbb{R}^n$, we write $L^{p,w}(E)$ by the weighted Lebesgue space over E that collects any functions f defined on E such that $\|f\|_{L^{p,w}(E)}$ is finite where

$$\|f\|_{L^{p,w}(E)} = \left(\int_E |f(x)|^p w(x)dx\right)^{\frac{1}{p}}.$$

If $E = \mathbb{R}^n$, we write $L^{p,w} = L^{p,w}(E) = L^{p,w}(\mathbb{R}^n)$.

Next, we give some definitions used in this paper.

Definition 2.1. [12] (A_p weight) Let $1 \leq p < \infty$. For $1 < p < \infty$, we define A_p as a set of all weights w on $\mathbb{R}^n$ for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(a, r)|}\int_{B(a,r)} w(x)dx\right)\left(\frac{1}{|B(a, r)|}\int_{B(a,r)} w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leq C$$

for all balls $B(a, r)$ in $\mathbb{R}^n$. For $p = 1$, we define A_1 as a set of all weights w for which there exists a constant $C > 0$ such that

$$\frac{1}{|B(a, r)|}\int_{B(a,r)} w(x)dx \leq C\|w\|_{L^\infty(B(a,r))} \tag{2.1}$$

for all balls $B(a, r)$ in $\mathbb{R}^n$ where

$$\|w\|_{L^\infty(B(a,r))} = \operatorname*{ess\,sup}_{x\in B(a,r)} w(x) = \inf\{M \geq 0 : |\{x \in B(a, r) : w(x) > M\}| = 0\}.$$

Remark 1. The last inequality (2.1) is equivalent to say that

$$\left(\frac{1}{|B(a, r)|}\int_{B(a,r)} w(x)dx\right) \cdot \|w^{-1}\|_{L^\infty(B(a,r))} \leq C$$

for all balls $B(a, r)$ in $\mathbb{R}^n$.

Theorem 2.1. *[12], [13] For $1 \leq p < \infty$ and $w \in A_p$, there exists $C > 0$ such that*

$$\frac{w(B)}{w(E)} \leq C \left(\frac{|B|}{|E|} \right)^p$$

for all balls B and measurable sets $E \subseteq B$ where $w(B) = \int_B w(x)dx$.

Definition 2.2. [13], [14] Let $1 < p < q < \infty$ and p' satisfies $1/p + 1/p' = 1$. We denote $A_{p,q}$ the collection of all weight functions w satisfying

$$\left(\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)^q dx \right)^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)^{-p'} dx \right)^{\frac{1}{p'}} \leq C$$

for all $(a, r) \in \mathbb{R}^n \times (0, \infty)$ where C is a constant independet of a and r. For $p = 1$ and $q > 1$, we denote $A_{1,q}$ the collection of weight functions w for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B(a,r)|} \int_{B(a,r)} w(x)^q dx \right)^{1/q} \leq C \|w\|_{L^\infty(B(a,r))}$$

for all $(a, r) \in \mathbb{R}^n \times (0, \infty)$.

Theorem 2.2. *Let $1 \leq p < q < \infty$. Then $w \in A_{p,q}$ if and only if $w^q \in A_{q/p'+1}$. Moreover, if $w \in A_{p,q}$, then $w^p \in A_p$ and $w^q \in A_q$.*

Definition 2.3. [15], [16] $BMO = BMO(\mathbb{R}^n)$ is set of all locally integrable functions b such that

$$\|b\|_* = \sup_{B=B(a,r)} \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty$$

where

$$b_B = \frac{1}{|B|} \int_B b(y) dy.$$

Definition 2.4. (Fractional Integrals) For $0 < \alpha < n$, the Riesz potential or the fractional integrals operator I_α is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n$$

for suitable functions f on $\mathbb{R}^n$.

Definition 2.5. (Commutator) Let b a locally integrable function defined on $\mathbb{R}^n$. For linear operator T, we define the commutator of T by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x), \quad x \in \mathbb{R}^n.$$

We rewrite the following results for $[b, T]$ and $[b, I_\alpha]$ on weighted Lebesgue spaces as well as the properties of $b \in BMO$. Note that the first following theorem based on the results in [17].

Theorem 2.3. *[9] Let $b \in BMO$ and I_α be the fractional integrals. If $0 < \alpha < n$, $1 < p < n/\alpha$, and $w \in A_p$, then $[b, I_\alpha]$ is bounded from L^{p,w^p} to L^{q,w^q}.*

Theorem 2.4. *[18] Let $b \in BMO$. Then, there is a constant $C > 0$ such that for all ball $B = B(a, r)$ in $\mathbb{R}^n$ and $j \in \mathbb{R}^n$,*

$$|b_{2^{j+1}B} - b_B| \leq C \cdot (j + 1) \|b\|_*.$$

Theorem 2.5. *[19] Let $b \in BMO$ and $1 \leq p < \infty$. Then, there is a constant $C > 0$ such that for all $B = B(a, r)$ in $\mathbb{R}^n$ and $w \in A_p$,*

$$\left(\int_B |b(y) - b_B|^p w(y) dy \right)^{\frac{1}{p}} \leq C \|b\|_* w(B)^{\frac{1}{p}}.$$

We now present the definition of generalized weighted Morrey spaces, generalized weighted weak Morrey spaces, and generalized weighted space of Log-type. These spaces will become the spaces of our main interest in this paper.

Definition 2.6. (Generalized Weighted Morrey Space) Let $1 \leq p < \infty, w \in A_p$, and ψ be a positive function on $\mathbb{R}^n \times (0, \infty)$. $\mathcal{M}_\psi^{p,w} = \mathcal{M}_\psi^{p,w}(\mathbb{R}^n)$ is set of all measurable functions f such that the norm $\|f\|_{\mathcal{M}_\psi^{p,w}}$ is finite where

$$\begin{aligned}
\|f\|_{\mathcal{M}_\psi^{p,w}} &= \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\psi(a,r)} \left(\frac{1}{w(B(a,r))} \int_{(B(a,r))} |f(x)|^p w(x) dx \right)^{1/p} \\
&= \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\psi(a,r)} \frac{1}{w(B(a,r))^{\frac{1}{p}}} \|f\|_{L^{p,w}(B(a,r))}.
\end{aligned}$$

3. Some Lemmas

Before proving the main results, we provide some lemmas which are very useful for that.

Lemma 3.1. *Let $1 < p < \infty$ and $w \in A_p$. If $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$ and $w \in A_{p,q}$, then*

$$\frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}} \int_{B(a,s)} |f(y)| dy \leq C_2 \frac{1}{w^q(B(a,s))^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(B(a,s))}$$

for all $a \in \mathbb{R}^n$ and $r > 0$, where C_2 is a constant that is independent of a and r.

Proof. Let $0 < \alpha < n, 1 < p < n/\alpha, 1/q = 1/p - \alpha/n$ and $w \in A_{p,q}$. By using Hölder's inequality,

$$\begin{aligned}
\frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}} \int_{B(a,s)} |f(y)| dy &= \frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}} \int_{B(a,s)} \frac{|f(y)| w(y)}{w(y)} dy \\
&\leq \frac{1}{|B(a,s)|^{1-\frac{\alpha}{n}}} \left(\int_{B(a,s)} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \left(\int_{B(a,s)} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \\
&= \frac{1}{w^q(B(a,s))^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(B(a,s))} \left(\frac{1}{|B(a,s)|} \int_{B(a,s)} w(y)^q dy \right)^{\frac{1}{q}} \\
&\cdot \left(\frac{1}{|B(a,s)|} \int_{B(a,s)} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \leq C \frac{1}{w^q(B(a,s))^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(B(a,s))}.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let φ a nonnegative function on $\mathbb{R}^n \times (0, \infty)$ such that the map $r \mapsto \varphi(a, r)$ is increasing for all $a \in \mathbb{R}^n$. Then, for each $1 \leq p < \infty, w \in A_p$, and the ball $B(a, r)$ we have*

$$\varphi(a,r) \leq C w(B(a,r))^{\frac{1}{p}} \int_r^\infty \frac{1}{w(B(a,s))^{\frac{1}{p}}} \varphi(a,s) \frac{ds}{s}$$

where $C > 0$ is independent of $a \in \mathbb{R}^n$ and $r > 0$.

The proof of Lemma 3.2 using the basic properties of integral as well as Theorem 2.1, so we omit here. Next, let B be a fixed ball in $\mathbb{R}^n$. For measurable function f on $\mathbb{R}^n$, we write $f = \sum_{k=0}^{\infty} f_k$ where $f_0 = f \cdot \mathcal{X}_{2B}$ and $f_k = f \cdot \mathcal{X}_{2^{k+1}B \setminus 2^k B}$ for $k > 0$. We have the following lemma.

Lemma 3.3. *Let $0 < \alpha < n, 1 < p < n/\alpha$, and $w \in A_p$. Then, there is a constant $D > 0$ such that*

$$\|[b, I_\alpha] f_k\|_{L^{q,w^q}(B(a,r))} \leq D\|b\|_* w^q(B(a,r))^{\frac{1}{q}} \frac{k+3}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}, \quad f \in L^{p,w^p}_{loc}$$

for any $k \in \mathbb{N}$.

Proof of Corollary 1.1. Let $B = B(a,r)$. By the definition,

$$[b, I_\alpha] f_k(x) = b(x) I_\alpha f_k(x) - I_\alpha(b f_k)(x), \quad x \in \mathbb{R}^n.$$

It then implies that

$$|[b, I_\alpha] f_k(x)| \leq |b(x) - b_B||I_\alpha f_k(x)| + |I_\alpha([b_B - b_{2^{k+1}B}] f_k)(x)| + |I_\alpha([b_{2^{k+1}B} - b] f_k)(x)|.$$

We shall estimate the three terms on the right side in the as inequality. For the first term, by Lemma 3.1 we have that

$$\begin{aligned}
|b(x) - b_B| \cdot |I_\alpha f_k(x)| &\leq |b(x) - b_B| \int_{\mathbb{R}^n} \frac{|f_k(x)|}{|x-y|^{n-\alpha}} dy \\
&\leq C|b(x) - b_B| \int_{2^{k+1} \setminus 2^k B} \frac{|f(y)|}{|a-y|^{n-\alpha}} dy \\
&\leq C|b(x) - b_B| \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |f(y)| dy \\
&\leq C|b(x) - b_B| \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}.
\end{aligned}$$

Theorem 2.5 then implies that

$$\|(b - b_B) I_\alpha f_k\|_{L^{p,w^p}(B)} \leq C\|b\|_* w^p(B)^{\frac{1}{p}} \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}}.$$

For the second term, by Theorem 3.2 and 2.4, we have

$$\begin{aligned}
|I_\alpha([b_B - b_{2^{k+1}B}] f_k)(x)| &\leq \int_{\mathbb{R}^n} \frac{|b_B - b_{2^{k+1}B}| \cdot |f_k(y)|}{|x-y|^{n-\alpha}} dy \\
&\leq C|b_B - b_{2^{k+1}B}| \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |f(y)| dy \\
&\leq C\|b\|_*(k+1) \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |f(y)| dy \\
&\leq C\|b\|_*(k+1) \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}.
\end{aligned}$$

Since $w \in A_{p,q}$, we have $w^{-p'} \in A_{p'}$. Hence, for the third term, Lemma 3.1, Theorem 2.5, and Holder's inequality then imply

$$\begin{aligned}
&|I_\alpha([b_{2^{k+1}} - b]f_k)(x)| \\
&\leq \int_{\mathbb{R}^n} \frac{|b_{2^{k+1}B} - b(y)| \cdot |f_k(y)|}{|x-y|^{n-\alpha}} dy \\
&\leq C \int_{2^{k+1}B \setminus 2^k B} \frac{|b_{2^{k+1}B} - b(y)| \cdot f(y)|}{|a-y|^{n-\alpha}} dy \\
&\leq C \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \int_{2^{k+1}B} |b_{2^{k+1}B} - b(y)| \cdot |f(y)| dy \\
&\leq \frac{1}{|2^{j+1}B|^{1-\frac{\alpha}{n}}} \left(\int_{2^{j+1}B} |f(y)|^p w(y)^p dy \right)^{\frac{1}{p}} \cdot \left(\int_{2^{j+1}B} |b_{2^{j+1}B} - b(y)|^{p'} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \\
&\leq C \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \|f\|_{L^{p,w^p}(2^{k+1}B)} \|b\|_* w^{-p'}(2^{k+1}B)^{\frac{1}{p'}} \\
&= C\|b\|_* \frac{1}{w^p(2^{k+1}B)^{\frac{1}{p}}} \|f\|_{L^{p,w^p}(2^{k+1}B)} \frac{1}{|2^{k+1}B|^{1-\frac{\alpha}{n}}} \left(\int_{2^{k+1}B} w(y)^{-p'} dy \right)^{\frac{1}{p'}} \left(\int_{2^{k+1}B} w(y)^q dy \right)^{\frac{1}{q}} \\
&= C\|b\|_* \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} w(y)^q dy \right)^{\frac{1}{q}} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} w(y)^{-p'} dy \right)^{-p'} \\
&\leq C\|b\|_* \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}
\end{aligned}$$

and

$$\|I_\alpha([b_{2^{k+1}B} - b]f_k)\|_{L^{p,w^p}(B)} \leq C\|b\|_* w^p(B)^{\frac{1}{p}} \frac{1}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}.$$

Therefore,

$$\|[b, I_\alpha]f\|_{L^{p,w^p}(B)} \leq C\|b\|_* w^p(B)^{\frac{1}{p}} \frac{k+3}{w^q(2^{k+1}B)^{\frac{1}{q}}} \|f\|_{L^{p,w^p}(2^{k+1}B)}.$$

It then immediately completes the proof of Theorem 3.3. □

4. Proof of the Main Results

We prove our main results in this section.

Proof of Theorem 1.1. Let $a \in \mathbb{R}^n$ and $r > 0$. We write

$$f = \sum_{k=0}^{\infty} f_k$$

where $f_0 = f \cdot \mathcal{X}_{2B}$ and $f_k = f \cdot \mathcal{X}_{2^{k+1}B \setminus 2^k B}$ for $k \in \mathbb{N}$. By Theorem 2.3, we have

$$\|[b, I_\alpha]f_0\|_{L^{p,w^p}(B)} \leq \|[b, I_\alpha]f_0\|_{L^{p,w^p}} \leq C\|b\|_* \|f_0\|_{L^{p,w^p}} = C\|b\|_* \|f\|_{L^{p,w^p}(2B)}.$$

By Lemma 3.2, we have

$$\|[b, I_\alpha]f_0\|_{L^{p,w^p}(B)} \leq C\|b\|_* w^p(2B)^{\frac{1}{p}} \int_{2r}^{\infty} w^p(B(a,s))^{-\frac{1}{p}} \|f\|_{L^{p,w^p}(B(a,s))} \frac{ds}{s}.$$

Then, the assumption implies that

$$\begin{aligned}
\|[b,I_\alpha]f_0\|_{\mathcal{M}^{q,w^q}_{\psi_2}} &= \sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{1}{w^q(B(a,r))^{\frac{1}{q}}}\|[b,I_\alpha]f_0\|_{L^{q,w^q}(B(a,r))}\\
&\le C\|b\|_*\sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{w^p(B(a,2r))^{\frac{1}{p}}}{w^q(B(a,r))^{\frac{1}{q}}}\int_{2r}^\infty w^p(B(a,s))^{-\frac{1}{p}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s}\\
&\le C\|b\|_*\sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{w^q(B(a,2r))^{\frac{1}{q}}}{w^q(B(a,r))^{\frac{1}{q}}}\|f\|_{\mathcal{M}^{p,w^p}_{\psi_1}}\int_r^\infty \psi_1(a,s)\frac{ds}{s}\\
&\le C\|b\|_*\|f\|_{\mathcal{M}^{p,w^p}_{\psi_2(a,r)}}\frac{1}{\psi_2(a,r)}\int_r^\infty \psi_1(a,s)\frac{ds}{s}.
\end{aligned}$$

For $k\in\mathbb{N}$, by Lemma 3.2 and 3.3 we have that

$$\begin{aligned}
\|[b,I_\alpha]f_k\|_{\mathcal{M}^{q,w^q}_{\psi_2}} &\le \sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{1}{w^q(B(a,r))^{\frac{1}{q}}}\|f\|_{L^{q,w^q}(B(a,r))}\\
&\le C\sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{1}{w^q(B(a,r))^{\frac{1}{q}}}\|b\|_* w(B(a,r))^{\frac{1}{q}}\frac{k+3}{w^q(2^{k+1}B)^{\frac{1}{q}}}\|f\|_{L^{p,w^p}(2^{k+1}B)}\\
&\le C\|b\|_*\sup_{a\in\mathbb{R}^n,r>0}\frac{1}{\psi_2(a,r)}\frac{k+3}{w^q(2^{k+1}B)^{\frac{1}{q}}}\|f\|_{L^{p,w^p}(2^{k+1}B)}\\
&\le C\|b\|_*\sup_{a\in\mathbb{R}^n,r>0}\frac{k+3}{\psi_2(a,r)}\int_{2^{k+1}r}^\infty\frac{1}{w^p(B(a,s))^{\frac{1}{p}}}\|f\|_{L^{p,w^p}(B(a,s))}\frac{ds}{s}\\
&\le C\|b\|_*\|f\|_{\mathcal{M}^{p,w^p}_{\psi_1}}\frac{k+3}{\psi_2(a,r)}\int_{2^{k+1}r}^\infty\psi_1(a,s)\frac{ds}{s}\le C\|b\|_*\|f\|_{\mathcal{M}^{p,w^p}_{\psi_1}}\frac{k+3}{2^{k+1}}.
\end{aligned}$$

It the implies that

$$\|[b,I_\alpha]f_k\|_{\mathcal{M}^{q,w^q}_{\psi_2}}\le C\frac{k+3}{2^{k+1}}\|b\|_*\|f\|_{\mathcal{M}^{p,w^p}_{\psi_1}},\quad f\in\mathcal{M}^{p,w^p}_{\psi_1},k\in\mathbb{N}$$

and

$$\|[b,I_\alpha]f\|_{\mathcal{M}^{q,w^q}_{\psi_2}}\le C\|b\|_*\|f\|_{\mathcal{M}^{p,w^p}_{\psi_1}},\quad f\in\mathcal{M}^{p,w^p}_{\psi_1}.$$

It completes the proof of Theorem 1.1. □

Proof of Corollary 1.1. Suppose that there is a constant $C>0$ and $\beta>0$ such that

$$\psi_1(a,r)s^\beta\le C\psi_1(a,r)r^\beta,\quad (a,r)\in\mathbb{R}^n\times(0,+\infty).$$

Then, for all $\lambda\ge 2$ and $a\in\mathbb{R}^n$ we have that

$$\int_{\lambda r}^\infty\psi_1(a,s)\frac{ds}{s}=\int_{\lambda r}^\infty\psi_1(a,s)s^\beta\frac{ds}{s^{\beta+1}}\le\psi_1(a,r)r^\beta\int_{\lambda r}\frac{ds}{s^{\beta+1}}\le\psi_2(a,r)r^\beta\frac{1}{\alpha}\frac{1}{\lambda^\beta}\frac{1}{r^\beta}=\frac{1}{\lambda^\beta}\psi_1(a,r).$$

Then, the conclusion follows immediately from Theorem 1.1. □

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