J. DIFERENSIAL/e-ISSN:2775-9644
Vol.8(1),2026, 56–73
doi:https://doi.org/10.35508/jd.v8i1.20714

RESEARCH ARTICLE

Perturbed Akbari-Ganji Method for the Solution of Singular Multi-Order Fractional Differential Equations

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Received: 22 February 2025; Revised: 17 January 2026; Accepted: 19 February 2026; Published: 1 April 2026.

Abstract:
Differential equations, which involve derivatives, are fundamental in describing various physical and engineering phenomena. Newton's second law of motion provides a basic example, which illustrates how force, mass, and acceleration relate through differential equations. These equations are widely used in science and engineering to model real-world systems. Fractional differential equations extend this concept by incorporating non-integer derivatives, allowing for a more generalized approach to complex problems. Multi-order fractional equations involve multiple fractional derivatives, while singular fractional equations contain terms that become undefined at specific points. We aim to explore the significance of fractional and singular fractional differential equations in mathematical modeling, highlighting their applications in capturing intricate behaviours across different fields and our results emphasize the broader applicability of these equations in solving advanced problems in physics, engineering, and applied sciences.

Keywords: Applied Sciences, Differential Equations, Fractional Derivatives, Mathematical Modeling, Singular Equations

1. Introduction

A differential equation has been defined as any equation which contains derivatives, either ordinary or partial derivatives. [1] showed a very common differential equation with the Newton's second law of motion which states that "If an object of mass m is moving with acceleration a and being acted upon by a force F, then F can be expressed as $F = ma$. Since F is a function of time, t velocity v and also acceleration a is the derivative of velocity with respect to time, then we write $a = \frac{dv}{dt}$ so that $F = ma$ can be rewritten as $F(t, v) = m\frac{dv}{dt}$ and this is a simple differential equation; i.e., $(m\frac{dv}{dt} = F(t, v))$. This points to the fact that differential equations result from our every day life. According to [2, 3] most differential equations results from physical phenomenal in connection with numerous problems that are encountered in the different fields of science and engineering.
On the other hand, fractional differential equations are differential equations with non integer differentials. For example,

$$y^{(n)}(x) + y(x) = f(x) \tag{1.1}$$

is a differential equation which can also be expressed in the form

$$D^{\alpha}\left(y(x)\right)=f(x) \tag{1.2}$$

is a fractional differential equation, and

$$D^{\alpha}\left(y(x)\right)+D^{\beta}\left(y(x)\right)+\left(y(x)\right)=f(x) \tag{1.3}$$

is multi order fractional differential equation, α and β are the non integer derivatives. Further more, a singular fractional differential equation is of the form

$$D^{\alpha}\left(y(t)\right)+\frac{k}{x^{\alpha-\beta}}D^{\beta}\left(y(t)\right)+\frac{k}{x^{\alpha-2}}\left(y(t)\right)=f(t) \tag{1.4}$$

which are singular at the points (coefficient) of $x=0$. α and β denote the orders of fractional differentiation in a fractional differential equation. Unlike classical derivatives of integer order, α and β are generally non-integer real numbers. They extend the concept of differentiation to account for memory and hereditary effects in dynamical systems. In equations involving multiple fractional derivatives, α and β represent different derivative orders, allowing the model to capture dynamics occurring at different temporal scales as discussed in [4–6].

2. Material and Method

Here we describe the method; the step by step procedure of the proposed method to be applied. We consider the class of Singular Multi-order Fractional Differential Equations.

$$D^{\alpha}\left(y(t)\right)+\frac{k}{x^{\alpha-\beta}}D^{\beta}\left(y(t)\right)+\frac{k}{x^{\alpha-2}}\left(y(t)\right)-f(t)=0 \tag{2.5}$$

subject to initial conditions

$$y(0)=A, \quad y'(0)=B \tag{2.6}$$

Equation (2.5) is slightly perturbed with the initial condition to obtain

$$D^{\alpha}\left(y(t)\right)+\frac{k}{x^{\alpha-\beta}}D^{\beta}\left(y(t)\right)+\frac{k}{x^{\alpha-2}}\left(y(t)\right)-f(t)+H_n(t)=0 \tag{2.7}$$

where

$$\begin{aligned} H_n(t) &= \sum_{(r=1)}^{n}\tau_r L_{(N-r-1)}(t) \\ &\text{or} \\ H_n(t) &= \sum_{(r=1)}^{n}\tau_r T_{(N-r-1)}(t) \end{aligned} \tag{2.8}$$

and $L_{(N-r-1)}(t)$ and $T_{(N-r-1)}(t)$, stand for Legendre and Chebyshev polynomials respectively and are called the perturbation terms.

To solve equation (1.1) subject to (1.2) we assume a trial solution of the form:

$$\begin{aligned} y_N(xt) &= \sum_{i=0}^{N} c_i L_i(t) \\ &\text{or} \\ y_N(xt) &= \sum_{i=0}^{N} c_i T_i(t) \end{aligned} \tag{2.9}$$

Now, when the trial solution is in Legendre polynomial, we perturb with Chebychev and vice versa so that equation (1.1) becomes

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \tfrac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \tfrac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(t) = 0 \tag{2.10}$$

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \tfrac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \tfrac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(t) = 0 \tag{2.11}$$

Evaluating equation (2.9) at $x = 0$ the initial conditions, gives

$$\begin{aligned} y_N(0) &= \sum_{i=0}^{N} c_i L_i(0) = A \\ y'_N(0) &= \sum_{i=0}^{N} c_i L'_i(0) = B \end{aligned} \tag{2.12}$$

$$\begin{aligned} y_N(0) &= \sum_{i=0}^{N} c_i T_i(0) = A \\ y'_N(0) &= \sum_{i=0}^{N} c_i T'_i(0) = B \end{aligned} \tag{2.13}$$

Now, discussing the remaining steps as follows;

2.1. Trial solutions in Legendre polynomial

Here, we take the trial solutions in Legendre polynomial and perturbed it with Chebyshev polynomial. We compute the derivatives of (2.10) with respect to α and β for α, $\alpha - 1$, $\alpha - 2$, $\cdots$ up to $\alpha - (n-2)$ and β, $\beta - 1$, $\beta - 2$, $\cdots$ up to $\beta - (n-2)$, we get:

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(t) = 0 \tag{2.14}$$

$$D^{\alpha-1}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-1}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(t) = 0 \tag{2.15}$$

$$D^{\alpha-2}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-2}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(t) = 0 \tag{2.16}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$D^{\alpha-(n-2)}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-(n-2)}\left(\sum_{i=0}^{N} c_i L_i(t)\right) + \frac{k}{x^{\alpha}}\left(\sum_{i=0}^{N} c_i L_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(t) = 0 \tag{2.17}$$

evaluating equations (2.14) to (2.17) at the initial condition; $t = 0$ we get

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(0)\right) - f(0) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(0) = 0 \tag{2.18}$$

$$D^{\alpha-1}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-1}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(0)\right) - f(0) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(0) = 0 \tag{2.19}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$D^{\alpha-(n-2)}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-(n-2)}\left(\sum_{i=0}^{N} c_i L_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i L_i(0)\right) - f(0) + \sum_{(r=1)}^{n} \tau_r T_{(N-r-1)}(0) = 0 \tag{2.20}$$

We have n equations altogether with n number of equations equal to the number of unknown constants in the assumed solution. After the evaluation of the equations at initial conditions, $t = 0$ we exit from differential equations to solving the set of algebraic system of equations. Solving equations (2.13), (2.14) and the new resulting equations from (2.18) to (2.20) using the Gaussian elimination method or any mathematical software, the constant coefficients are obtained as seen in [7]. The values of the constants are substituted back into the assumed solution (2.13) which yields the approximate solution for the Legendre polynomial [8, 9].

Perturbing the Chebyshev polynomial with the Legendre polynomial

Here, we repeat the process, now by perturbing the Chebyshev polynomial with the Legendre polynomial. Consider the class of Singular Multi-order Fractional Differential Equations (2.9) and following the same process. Computing the derivatives of (2.15) with respect to α and β for α, $\alpha - 1$, $\alpha - 2, \cdots$ up to $\alpha - (n-2)$, we get:

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(t) = 0 \tag{2.21}$$

$$D^{\alpha-1}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-1}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(t) = 0 \tag{2.22}$$

$$D^{\alpha-2}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-2}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(t) = 0 \tag{2.23}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$D^{\alpha-(n-2)}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-(n-2)}\left(\sum_{i=0}^{N} c_i T_i(t)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(t)\right) - f(t) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(t) = 0 \tag{2.24}$$

Now evaluating equations (2.21) to (2.24) at the initial condition; $t = ic = 0$ we get

$$D^{\alpha}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(0)\right) - f(0) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(0) = 0 \tag{2.25}$$

$$D^{\alpha-1}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-1}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(0)\right) - f(0) + \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(0) = 0 \tag{2.26}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$D^{\alpha-(n-2)}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-\beta}} D^{\beta-(n-2)}\left(\sum_{i=0}^{N} c_i T_i(0)\right) + \frac{k}{x^{\alpha-2}}\left(\sum_{i=0}^{N} c_i T_i(0)\right) - f(0) + K \tag{2.27}$$

where

$$K = \sum_{(r=1)}^{n} \tau_r L_{(N-r-1)}(0) = 0.$$

So, we have n equations altogether with n number of equations equal to the number of unknown constants in the assumed solution. After the evaluation of the equations at initial conditions, $t = 0$ we exit from differential equations to solving the set of algebraic system of equations. Solving equations (2.13), (2.14) and the new resulting equations from (2.25)- (2.27) using Gaussian elimination method or any mathematical software, the constant coefficients are obtained [10–12]. Values of the constants are substituted back into the assumed solution (2.13) which yields the approximate solution for Chebyshev polynomial.

3. Numerical Examples

3.1. Example 1

Consider the class of Singular Multi order Fractional Differential Equations.

$$y^{\alpha}(x) + \frac{2}{x} y'(x) + y(x) = 6 + 12x + x^2 + x^3 \qquad 0 \le x \le 1 \tag{3.28}$$

subject to initial conditions (ic)

$$y(0) = y'(0) = 0 \tag{3.29}$$

Exact solution is $x^2 + x^3$ as also discussed in [13]. Equation (3.28) is slightly perturbed as,

$$y^{\alpha}(x) + \frac{2}{x} y'(x) + y(x) - 6 - 12x - x^2 - x^3 + H_n T(x) = 0 \tag{3.30}$$

where $H_n T(x)$ is defined in equation (2.8). Assume a trial solution,

$$y_N(x) = \sum_{i=0}^{N} a_i L_i(x) = a_0 + a_1\left(2x - 1\right) + a_2\left(6x^2 - 6x + 1\right) + a_3\left(20x^3 - 30x^2 + 12x - 1\right) + a_4\left(70x^4 - 140x^3 + 90x^2 - 20x + 1\right) \tag{3.31}$$

To solve the problem at $\alpha = 1.5$ and $\alpha = 2.5$, check the exactness as deduced from above. Then, when $\alpha = 1.5$ and D^{α} on equation (3.28). These values are deliberately chosen to represent non-integer derivative orders lying between successive integers, thereby highlighting the transition from classical to fractional dynamics.In particular, $\alpha = 1.5$ captures intermediate memory effects between first- and second-order systems, while $\alpha = 2.5$ reflects stronger hereditary behaviour beyond second-order dynamics. The operator D^{α} is employed to demonstrate the capability of fractional derivatives to model processes with memory, which cannot be adequately described by integer-order derivatives alone. Thus, the selected values of α (and similarly β, when applicable) allow assessment of the model's flexibility and accuracy across different fractional orders. It is obtained that,

$$\begin{aligned} y(x) \quad = \quad & 256\,x^4a_4 + 64\,x^3a_3 - 448\,x^3a_4 + 16\,x^2a_2 - 80\,x^2a_3 + 240\,x^2a_4 \\ & +4\,a_1x - 12\,xa_2 + 24\,xa_3 - 40\,xa_4 + a_0 - a_1 + a_2 - a_3 + a_4 \end{aligned} \tag{3.32}$$

Substituting equation (3.31) into the problem accordingly and evaluating it result, we have equation (3.33) below.

$$\begin{aligned} y(x) \quad = \quad & 256320\,a_0a_2 + 2 + 3.141776845625\,x\tau_2{}^2 + a_0a_4 - 5.333472196\,a_0x^{2.0} - 2\,a_1a_2 + 2\,a_1a_3 \\ & -2\,a_1a_4 + 305.333472196\,a_1x^{2.0} - 2\,a_2a_3 + 24528\,a_2a_4 - 5.3363472196\,a_2x^{2.0} - 2\,a_3a_4 \\ & +5.333472196\,a_3x^{2.0} - 5.333472196\,a_4x^{2.0} - 75.62863574\,x^{4.500000000}\tau_1 + 642.8434038 \\ & x^{3.500000000}\tau_1 - 9.4535872179468\,x^{2.500000000}\tau_1 - 18.90715893\,x^{3.500000000}\tau_2 + 9.453579468 \\ & x^{2.500000000}\tau_2 + 201.072404300\,x^5\tau_1{}^2 - 3418.230800\,x^4\tau_1{}^2 + 14577.74900\,x^3\tau_1{}^2 \\ & -427.2788500\,x^2\tau_1{}^2 + 3.14175625\,x\tau_1{}^2 + 12.56702500\,x^3\tau_2{}^2 - 12.56702500\,x^2\tau_2{}^2 + 65536 \\ & x^8a_4{}^2 - 229376\,x^7a_4{}^2 + 323584\,x^6a_4{}^2 - 235520\,x^5a_4{}^2 + 93952\,x^4a_4{}^2 - 1365.368882 \\ & x^{6.0}a_4 + 4096\,x^6a_3{}^2 - 10240\,x^5a_3{}^2 + 9472\,x^4a_3{}^2 - 3968\,x^3a_3{}^2 - 341.3422206\,x^{5.0}a_3 \\ & -20096\,x^3a_4{}^2 + 2389.395544\,x^{5.0}a_4 + 256\,x^4a_2{}^2 - 384\,x^3a_2{}^2 + 176\,x^2a_2{}^2 - 85.33555514 \\ & x^{4.0}a_2 + 736\,x^2a_3{}^2 + 426.6777756\,x^{4.0}a_3 + 2080\,x^2a_4{}^2 - 1280.033327\,x^{4.0}a_4 + 16\,a_1{}^2x^2 \\ & -8\,a_1{}^2x - 21.33388878\,a_1x^{3.0} - 24\,xa_2{}^2 + 64.00166636\,x^{3.0}a_2 - 48\,xa_3{}^2 - 128.0033327 \\ & x^{3.0}a_3 - 80\,xa_4{}^2 + 213.3388878\,x^{3.0}a_4 - 2\,a_0a_1 + a_0{}^2 + 7.111481416\,x^{4.0} + a_1{}^2 + a_2{}^2 + a_3{}^2 \\ & +a_4{}^2 - 6097.4000\,x^{5/2}a_3\tau_1 + 453.7600\,x^{5/2}a_3\tau_2 + 480\,x^2a_4a_0 - 800\,x^2a_4a_1 + 10521.5600 \\ & x^{5/2}a_4\tau_1 - 1134.4000\,x^{5/2}a_4\tau_2 + 8\,a_1xa_0 + 32\,a_1xa_2 - 56\,a_1xa_3 + 88\,a_1xa_4 + 113.4400 \\ & a_1x^{7/2}\tau_1 - 992.6000\,a_1x^{5/2}\tau_1 + 255.2400\,a_1x^{3/2}\tau_1 + 28.3600\,a_1x^{5/2}\tau_2 - 21.2700 \end{aligned} \tag{3.33}$$

Equation (3.28) is integrated from (0 to 1), then we have equation (3.34) below as;

$$\begin{aligned} y(x) \quad = \quad & 0.6666666667\,a_0a_2 + 0.6666666667\,a_0a_3 + 0.4000000000\,a_0a_4 + 3.333333333\,a_1a_2 \\ & +1.733333333\,a_1a_3 + 1.46666667\,a_1a_4 + 4.133333\,a_2a_3 + 2.419047619\,a_2a_4 \\ & +4.704761905\,a_3a_4 + 2.0\,a_0a_1 - 62.62567458\,\tau_1\tau_2 - 85.95780952\,a_0\tau_1 \\ & +0.4726666667\,a_0\tau_2 + 2.183269841\,a_2\tau_2 - 65.17338284\,a_3\tau_1 + 0.8567181707\,a_4\tau_2 \\ & +1.180643579\,a_3\tau_2 - 48.89620979\,a_4\tau_1 - 110.5037374\,a_2\tau_1 - 158.6584444\,a_1\tau_1 \\ & +1.958190476\,a_1\tau_2 + 1.422296283 + a_0{}^2 - 1.500568168\,\tau_2 + 0.523260417\,\tau_2{}^2 \\ & -1.777824065\,a_0 - 2.844518503\,a_2 - 1.7778240\,a_3 - 1.269874277\,a_4 - 3.555648130\,a_1 \\ & +2.333333333\,a_1{}^2 + 2.8666667\,a_2{}^2 + 3.209523810\,a_3{}^2 + 3.4634923\,a_4{}^2 \\ & +126.4024060\,\tau_1 + 2853.447751\,\tau_1{}^2. \end{aligned} \tag{3.34}$$

Evaluating the partial derivative of equation (3.32) with respect to the variables, we have

$$y(x) = 0.50000000\,a_0 - 1.50000000\,a_1 - 4.50000000\,a_2 - 6.50000000\,a_3 - 8.70000000\,a_4 + 0.6666840245 = 0 \tag{3.35}$$

$$y(x) = -1.50000000\,a_0 + 5.166666666\,a_1 + 19.500000\,a_2 + 3.5000000\,a_3 + 2.36666667\,a_4 - 2.800072902 = 0 \tag{3.36}$$

$$y(x) = -4.500000000\,a_0 + 9.50000000\,a_1 + 10.9000000\,a_2 + 8.5000000\,a_3 + 2.3000000\,a_4 - 1.26703812 = 0 \tag{3.37}$$

$$y(x) = 7.7857142\,a_3 + 0.7000000\,a_4 - 6.500000000\,a_0 + 1.50000000\,a_1 + 8.5000000\,a_2 - 28.70550928 = 0 \tag{3.38}$$

$$y(x) = 10.7000000\,a_3 + 1.655556\,a_4 - 8.700000000\,a_0 + 5.36666667\,a_1 + 2.3000000\,a_2 - 4.00109360 = 0. \tag{3.39}$$

Solving equations (3.35) to (3.40) using maple 18, we have

$$y(x) = a_0 = 0.3333419922, a_1 = 0.5000130399, a_2 = 0.1666709810,$$
$$a_3 = 3.637303510 \times 10^{-8}, a_4 = -2.734845470 \times 10^{-8}.$$

Substituting these values into equation (3.31), we have the approximate solution

$$y(x) = -0.1170779082e^{-5}x^4 + 1.004259256655x^3 + 1.0434973200x^2 + 0.121483195e^{-2}x - 0.40051098e^{-2} \tag{3.40}$$

For $\alpha = 2.5$, solving Problem 1 taking $\alpha = 2.5$ following the same procedure. We apply D^α for $\alpha = 2.5$ on equation (3.30), and after the necessary simplification and integrated from 0 to 1. We have,

$$\begin{aligned} y(x) \;=\; & 0.6666666667\,a_0a_2 + 0.6666666667\,a_0a_3 + 0.4000000000\,a_0a_4 + 3.333333333\,a_1a_2 \\ & +1.73333\,a_1a_3 + 1.46666667\,a_1a_4 + 4.133333\,a_2a_3 + 2.419047619\,a_2a_4 \\ & +4.704761905\,a_3a_4 + 2.0\,a_0a_1 - 62.62567458\,\tau_1\tau_2 - 85.95780952\,a_0\tau_1 \\ & +0.4726666667\,a_0\tau_2 + 2.183269841\,a_2\tau_2 - 65.17338284\,a_3\tau_1 + 0.8567181707\,a_4\tau_2 \\ & +1.180643579\,a_3\tau_2 - 48.89620979\,a_4\tau_1 - 110.5037374\,a_2\tau_1 - 158.6584444\,a_1\tau_1 \\ & +1.958190476\,a_1\tau_2 + 1.422296283 + {a_0}^2 - 1.500568168\,\tau_2 + 0.523260417\,{\tau_2}^2 \\ & -1.777824065\,a_0 - 2.844518503\,a_2 - 1.7778240\,a_3 - 1.269874277\,a_4 - 3.555648130\,a_1 \\ & +2.333333333\,{a_1}^2 + 2.8666667\,{a_2}^2 + 3.209523810\,{a_3}^2 + 3.4634923\,{a_4}^2 \\ & +126.4024060\,\tau_1 + 2853.447751\,{\tau_1}^2. \end{aligned} \tag{3.41}$$

Evaluating the partial derivative of equation (3.40) with respect to the variables, we have

$$y(x) = a_0 = 1.436962430, a_1 = 0.4401128276, a_2 = 0.1537722618,$$
$$a_3 = 0.2083436194 \times 10^{-7}, a_4 = 0.1397634912 \times 10^{-8}$$

Substituting these values into equation (2.13), we have the approximate solution

$$y(x) = -0.1914391829e^{-5}x^4 + .9000004556244360x^3 + 1.000022333x^2 + 0.1177e^{-5}x - 1.304 \times 10^{-7} \tag{3.42}$$

The results are shown in Table 3.1 and Figure 3.1

Table 3.1: Error of Results for Example 1

x	Exact	$\alpha = 1.5$	Error	$\alpha = 2.5$	Error
0.0	0.0000000	-0.00400510	4.051e-08	-0.0000001	1.340e-07
0.1	0.0110000	0.02207872	1.279e-07	0.01000215	2.199e-07
0.2	0.0480000	0.07447037	3.470e-07	0.04000102	1.317e-06
0.3	0.1170000	0.15103577	3.136e-06	0.09000234	1.302e-06
0.4	0.2240000	0.24992183	5.922e-06	0.26000415	4.563e-06
0.5	0.3750000	0.36955644	1.156e-06	0.25000641	2.4912e-06
0.6	0.5760000	0.50864851	1.465e-05	0.3600093	3.317e-06
0.7	0.8330000	0.66618789	1.619e-05	0.49001273	1.740e-05
0.8	1.1520000	1.14144546	2.145e-05	1.64001663	1.653e-05
0.9	1.5390000	1.33973060	2.397e-05	1.81002104	2.184e-05
1.0	2.0000000	1.86360322	2.430e-05	1.90002602	2.021e-05

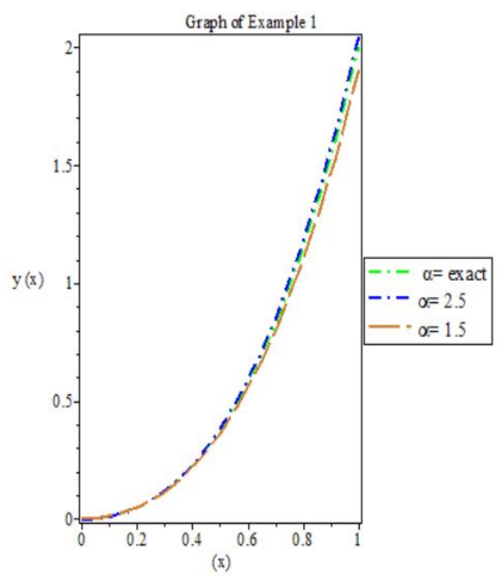

Figure 3.1: Graphical Representation of Error in Table 1

3.2. Example 2

Consider the class of Singular Multi order Fractional Differential Equations.

$$y^{\alpha}(x) + \frac{2}{x}y'(x) - 2(x^2+3)y(x) = 0 \qquad 0 \le x \le 1 \tag{3.43}$$

subject to initial conditions (ic).

$$y(0) = y'(0) = 0. \tag{3.44}$$

Exact solution is e^{x^2} also as seen in [14, 15]. Equation (3.43) is slightly perturbed as,

$$y^{\alpha}(x) + \frac{2}{x}y'(x) - 2(x^2+3)y(x) + H_nT(x) = 0 \tag{3.45}$$

where $H_nT(x)$ is defined in equation (2.9). Assume a trial solution,

$$\begin{aligned} y_N(x) = \sum_{i=0}^{N} c_iT_i(x) &= a0 + a_1\,(2\,x-1) + a_2\left(8\,x^2 - 8\,x + 1\right) + a_3\left(32\,x^3 - 48\,x^2 + 18\,x - 1\right) \\ &\quad + a_4\left(128\,x^4 - 256\,x^3 + 160\,x^2 - 32\,x + 1\right) \end{aligned} \tag{3.46}$$

In other to solve example 1 from above, when $\alpha = 1.5$ and $\alpha = 2.5$. Applying D^{α} for $\alpha = 1.5$ on equation (3.45), It is obtained that,

$$\begin{aligned} y(x) &= 896\,x^4a_4 + 160\,x^3a_3 - 1120\,x^3a_4 + 24\,x^2a_2 - 120\,x^2a_3 + 360\,x^2a_4 + 2\,a_1x - 6\,xa_2 \\ &\quad + 12\,xa_3 - 20\,xa_4 - 1/2\,a_0 + 1/2\,a_1 - 1/2\,a_2 + 1/2\,a_3 - 1/2\,a_4 \end{aligned} \tag{3.47}$$

Substituting equation (3.46) into the problem accordingly and evaluating, we have that

$$\begin{aligned} y(x) &= 2.666736098\,a_3x^{3.0} + 0.257843\,{a_4}^2 + 2.6667098\,a_4x^{3.0} + 802816\,x^8{a_4}^2 - 2007040\,x^7{a_4}^2 \\ &\quad - 842240\,x^5{a_4}^2 + 173504\,x^4{a_4}^2 - 4778.791088\,x^{7.0}a_4 + 2565600\,x^6{a_3}^2 - 38400\,x^5{a_3}^2 \\ &\quad + 18240\,x^4{a_3}^2 - 2720\,x^3{a_3}^2 - 853.3555514\,x^{6.0}a_3 - 13280\,x^3{a_4}^2 + 5973.488860\,l, x^{6.0}a_4 \\ &\quad + 576\,x^4{a_2}^2 - 288\,x^3{a_2}^2 + 12\,x^2{a_2}^2 - 128.0033327\,x^{5.0}a_2 + 24\,x^2{a_3}^2 + 640.0166636\,x^{5.0}a_3 \\ &\quad + 40\,x^2{a_4}^2 - 1920.049991\,x^{5.0}a_4 + 4\,{a_1}^2x^2 + 2\,{a_1}^2x - 10.66694439\,a_1x^{4.0} + 6\,x{a_2}^2 \\ &\quad + 32.00083318\,x^{4.0}a_2 + 12\,x{a_3}^2 - 64.00166636\,x^{4.0}a_3 + 20\,x{a_4}^2 + 106.6694439\,x^{4.0}a_4 \\ &\quad + 1/4\,{a_0}^2 - 1/2\,a_0a_1 + 1/2\,a_0a_2 - 1/2\,a_0a_3 + 1/2\,a_0a_4 + 2.666736098\,a_0x^{3.0} + 1/4\,{a_1}^2 \\ &\quad - 1/2\,a_1a_2 + 1/2\,a_1a_3 \end{aligned} \tag{3.48}$$

Equation (3.43) is integrated from (0 to 1), we have:

$$\begin{aligned} y(x) &= 712.8277778\,{a_4}^2 + 0.25000000\,{a_0}^2 - 1.500000000\,a_0a_1 - 4.500000000\,a_0a_2 \\ &\quad - 6.500000000\,a_0a_3 - 8.700000000\,a_0a_4 + 2.583333333\,{a_1}^2 + 19.50000000\,a_1a_2 \\ &\quad + 31.500000\,a_1a_3 + 42.36666667\,a_1a_4 + 50.45000000\,{a_2}^2 + 198.5000000\,a_2a_3 \\ &\quad + 282.3000000\,a_2a_4 + 239.3928571\,{a_3}^2 + 770.70000\,a_3a_4 + 1.015925917 + 0.66668245\,a_0 \\ &\quad - 2.800072902\,a_1 - 14.26703812\,a_2 - 28.70550928\,a_3 - 42.00109360\,a_4 \end{aligned} \tag{3.49}$$

Evaluating the partial derivative of equation (3.47) with respect to the variables, we have

$$\begin{aligned} y(x) &= 0.600000000\,a_0 - 1.45000000\,a_1 - 3.5000000\,a_2 - 6.600000000\,a_3 \\ &\quad - 8.400000000\,a_4 + 1.7666840245 = 0 \end{aligned} \tag{3.50}$$

$$\begin{aligned} y(x) &= -2.50000000\,a_0 + 4.466666666\,a_1 + 9.00500000\,a_2 + 3.66000000\,a_3 \\ &\quad +2.536666667\,a_4 - 2.568000722 = 0 \end{aligned} \tag{3.51}$$

$$\begin{aligned} y(x) &= -2.5600000000\,a_0 + 6.50000000\,a_1 + 5.19000000\,a_2 + 8.25000000\,a_3 \\ &\quad +2.3000000\,a_4 - 3.26703812 = 0 \end{aligned} \tag{3.52}$$

$$\begin{aligned} y(x) &= 5.71857142\,a_3 + 0.57000000\,a_4 - 2.500000000\,a_0 + 1.540000000\,a_1 \\ &\quad +8.56600000\,a_2 - 21.70550928 = 0 \end{aligned} \tag{3.53}$$

$$\begin{aligned} y(x) &= 8.754000000\,a_3 + 1.4655556\,a_4 - 5.700000000\,a_0 + 5.636666667\,a_1 \\ &\quad +2.453000000\,a_2 - 4.600109360 = 0 \end{aligned} \tag{3.54}$$

Solving equations (3.50) to (3.54) using maple 18, we have

$$\begin{aligned} y(x) &= a_0 = 0.1226853834, a_1 = 0.32324500013, a_2 = 0.1670981110, \\ &\quad a_3 = 2.351063730 \times 10^{-8}, a_4 = -1.421734845 \times 10^{-9} \end{aligned}$$

These values into equation (3.43),we have the approximate solution.

$$y(x) = 0.00001304 + 0.0001177\,x + 1.000022333\,x^2 + 1.000004556244360\,x^3 - 0.000000001914391829\,x^4 \tag{3.55}$$

For $\alpha = 2.5$, solving example 2 as given above while taking $\alpha = 2.5$ following the same procedure as discussed in [16]. We apply D^α for $\alpha = 2.5$ on equation (3.46), and after the necessary simplifications and integrated from 0 to 1. It is obtained as;

$$\begin{aligned} y(x) &= +198.50000\,a_2a_3 + 82.30\,a_2a_4 + 239.3928571\,{a_3}^2 + 70.700\,a_3a_4 + 1.015925917 \\ &\quad +0.66668245\,a_0 + 712.8277778\,{a_4}^2 + 0.25000\,{a_0}^2 - 1.500000\,a_0a_1 \\ &\quad -14.5000000\,a_0a_2 - 6.5000000\,a_0a_3 - 8.7000\,a_0a_4 + 2.583333333\,{a_1}^2 \\ &\quad +11.2345000\,a_1a_2 + 31.500\,a_1a_3 + 42.36666667\,a_1a_4 + 450.450000\,{a_2}^2 \\ &\quad -2.800072902\,a_1 - 14.26703812\,a_2 - 28.70550928\,a_3 - 42.00109360\,a_4 \end{aligned} \tag{3.56}$$

Evaluating the partial derivative of equation (3.55) with respect to the variables. It is obtained as,

$$\begin{aligned} y(x) &= a_0 = 1.351269624, a_1 = 0.124016128, a_2 = 0.1653757226, \\ &\quad a_3 = 0.5208347361 \times 10^{-8}, a_4 = 0.1533976234 \times 10^{-9} \end{aligned}$$

Substituting these values into equation (2.13), we have the approximate solution

$$y(x) = 0.0000002370 + 0.000050464\,x + 1.000053679\,x^2 + 1.0000084652136\,x^3 - 0.00000000181691421\,x^4 \tag{3.57}$$

The results of this is displayed in Figure 3.2 and Table 3.2 below.

Table 3.2: Error of Results for Example 2

x	Exact	$\alpha = 1.5$	Error	$\alpha = 2.5$	Error
0.0	1.000000	0.000000	1.6343e-07	0.000003	3.0646e-08
0.1	1.010050	1.010051	1.5406e-07	1.010049	1.5406e-07
0.2	1.040810	1.040815	2.7390e-07	1.040798	2.7390e-06
0.3	1.094174	1.094116	1.2886e-06	1.094201	1.2886e-06
0.4	1.173510	1.173381	3.5849e-05	1.173184	3.5849e-05
0.5	1.284025	1.282652	1.1521e-05	1.284621	1.1521e-05
0.6	1.433329	1.432175	1.3014e-05	1.432638	1.3014e-05
0.7	1.632316	1.630079	1.4008e-05	1.630178	1.4008e-04
0.8	1.896480	1.897841	1.64472e-04	1.897739	1.64472e-04
0.9	2.247907	2.235807	2.44011e-03	2.235927	2.44016e-03
1.0	2.718281	2.717813	1.38143e-02	2.716913	1.38147e-02

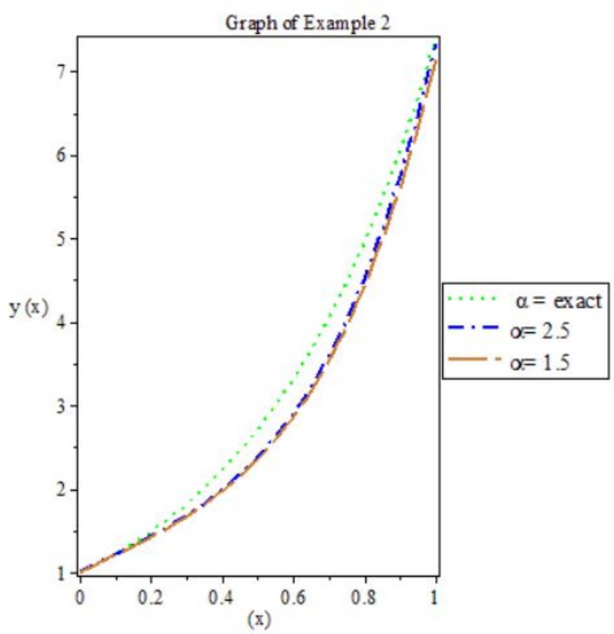

Figure 3.2: Graphical Representation of Error in Table 2

3.3. Example 3

Consider the class of Singular Multi-order Fractional Differential Equations.

$$y^{\alpha}(x) - \frac{1}{x^2}y'(x) - y(x) = 2\sqrt{x} + 8/3x^3/2 - \frac{2x+x}{x^2} - x - 1 \quad 0 \le x \le 1 \tag{3.58}$$

subject to initial conditions (ic).

$$y(0) = 0, \quad y'(0) = 1 \tag{3.59}$$

Exact solution is $x + x^2$. Equation (3.58) is slightly perturbed as,

$$y^{\alpha}(x) - \frac{1}{x^2}y'(x) - y(x) = 2\sqrt{x} + \frac{8}{3}x^{\frac{8}{3}} - \frac{2x+x}{x^2} - x - 1 + H_nT(x) = 0 \tag{3.60}$$

where $H_nT(x)$ is defined in equation (2.9). Assume a trial solution.

$$\begin{aligned} y_N(x) = \sum_{i=0}^{N} c_i L_i(x) \;&=\; a_0 + a_1(2x-1) + a_2\left(6x^2 - 6x + 1\right) + a_3\left(20x^3 - 30x^2 + 12x - 1\right) \\ &\quad + a_4\left(70x^4 - 140x^3 + 90x^2 - 20x + 1\right) \end{aligned} \tag{3.61}$$

We solved the problem at $\alpha = 1.5$ and $\alpha = 2.5$. Applying D^{α} for $\alpha = 1.5$ on equation (3.60), it is therefore obtained that,

$$\begin{aligned} y(x) \;=\;& 256x^4a_4 + 64x^3a_3 - 448x^3a_4 + 16x^2a_2 - 80x^2a_3 + 240x^2a_4 + 4a_1x - 12xa_2 + 24xa_3 \\ & -40xa_4 + a_0 - a_1 + a_2 - a_3 + a_4 \end{aligned} \tag{3.62}$$

Substituting equation (3.62) into the problem accordingly and evaluating, we have

$$\begin{aligned} y(x) \;=\;& 412921.6000\,x^{23/2}a_3a_4 - 12705.28000\,x^{\frac{25}{2}}a_4\tau_1 + 12705.280\,x^{\frac{25}{2}}a_4\tau_2 + 66702.7\,x^{23/2}a_4\tau_1 \\ & +1054.08\,x^{21/2}a_2a_4 - 1135534.4\,x^{21/2}a_3a_4 - 144862.88\,x^{21/2}a_4\tau_1 + 90865.440\,x^{21/2}a_4\tau_2 \\ & -2268.8\,x^{23/2}a_3\tau_1 + 228.80\,x^{23/2}a_3\tau_2 + 27395.76\,x^{19/2}a_1a_4 - 251.6\,x^{19/2}a_2a_4 \\ & +6157\,x^{19/2}a_3a_4 + 63.5\,x^{19/2}a_4\tau_1 - 7.0\,x^{19/2}a_4\tau_2 + 776.80\,x^{21/2}a_3\tau_1 - 808.0\,x^{21/2}a_3\tau_2 \\ & +208.9\,x^{17/2}a_0a_4 - 472.58\,x^{17/2}a_1a_4 + 899\,x^{17/2}a_2a_4 - 671.3\,x^{17/2}a_3a_4 - 987.0779\,x^{17/2}a_4\tau_1 \\ & +20419.2\,x^{19/2}a_2a_3 - 874589.36\,x^{19/2}a_3\tau_1 + 12081.36\,x^{19/2}a_3\tau_2 - 340.3200\,x^{21/2}a_2\tau_1 \\ & +340.320\,x^{21/2}a_2\tau_2 + 5104.800\,x^{17/2}a_1a_3 - 35733.60\,x^{17/2}a_2a_3 + 19958.35\,x^{17/2}a_3\tau_1 \\ & -7876.990\,x^{17/2}a_3\tau_2 - 148.890\,x^{13/2}a_3\tau_2 + 34262.330\,x^{13/2}a_4\tau_2 + 27651.27\,x^{15/2}a_0\tau_2 \\ & +14.180\,x^{11/2}a_0a_1 - 54342.540\,x^{11/2}a_0a_2 - 765.72\,x^{11/2}a_0a_3 + 5813\,x^{11/2}a_0a_4 \\ & +397.04\,x^{11/2}a_1a_2 - 95\,x^{11/2}a_1a_3 - 694.82\,x^{11/2}a_1a_4 + 642.50\,x^{11/2}a_2a_3 \\ & +56.72\,x^{11/2}a_2\tau_1 + 465.28\,x^{11/2}a_3a_4 - 141.8\,x^{11/2}a_3\tau_1 + 255.2\,x^{11/2}a_4\tau_1 \\ & +42.54\,x^{13/2}a_0\tau_1 - 21.27\,x^{13/2}a_0\tau_2 \end{aligned} \tag{3.63}$$

$$
\begin{aligned}
y(x) \;=\;& 72.0\,x^4a_2a_3+104.0\,x^4a_2a_4-2765.0\,x^3a_2a_3+2.0\,x^3a_2a_4-2.0\,x^3a_3a_4-295.6\,x^9a_3\tau_1\\
&+2446.18\,x^9a_4\tau_1-1018.30\,x^9a_4\tau_2+527.81\,x^9\tau_1\tau_2+150.80\,x^8a_0a_2-754.02\,x^8a_0a_3\\
&+274.666\,x^8a_0a_4150.80\,x^8a_0\tau_1+75.402\,x^8a_0\tau_2-452.412\,x^8a_1a_2+7389.41\,x^8a_1a_3\\
&-52269.232\,x^8a_1a_4+351.876\,x^8a_1\tau_1-657345.67025\,x^8a_1\tau_2-8187.50\,x^8a_2a_3\\
&+2029.94\,x^8a_2a_4-2714.477400\,x^8a_2\tau_1+1131.0322\,x^8a_2\tau_2-2733.904\,x^8a_3a_4\\
&+23223.86\,x^8a_3\tau_1-7917.225750\,x^8a_3\tau_2-137131.3768\,x^8a_4\tau_1+35263.07215\,x^8a_4\tau_2\\
&-175.9383500\,x^8\tau_1\tau_2+50.2681\,x^7a_0a_1-150.8043\,x^7a_0a_2+3317.6946\,x^7a_0a_3\\
&-25199.28300\,x^7a_0a_4+100.536\,x^7a_0\tau_1-25.13405760\,x^7a_0\tau_2+804860.26\,x^7a_1a_2\\
&-3824.0487\,x^7a_1a_3+16377.80\,x^7a_1a_4-150430\,x^7a_1\tau_1+257.1340\,x^7a_1\tau_2\\
&+2169.09\,x^7a_2a_3-188.2\,x^7a_2a_4+703.753\,x^{11}{\tau_1}^2-251.340\,x^{11}{\tau_2}^2\\
&-1759452.4129\,x^{10}{a_2}^2+776.21450\,x^{10}{a_3}^2+6185647.9652\,x^{10}{a_4}^2\\
&+879.69109\,x^{10}{\tau_1}^2+188.505\,x^{10}{\tau_2}^2 \qquad (3.64)
\end{aligned}
$$

Equation (3.61) is integrated from (0 to 1), we have;

$$
\begin{aligned}
y(x) \;=\;& 175.8530487\,a_1a_2+48.99512915\,a_0a_2+4961.882199\,a_2a_3+536.8186141\,a_1a_3\\
&+316.5417221\,a_0a_4+35486.64754\,a_3a_4+10080.68384\,a_2a_4+5.352654208\,a_0a_1\\
&+154.4817476\,a_0a_3-0.1045020835\,a_4\tau_2+0.0004882294\,\tau_1\tau_2-0.1812286852\,a_2\tau_1\\
&-0.3424305430\,a_2\tau_2-0.10215546\,a_3\tau_1-0.4496374580\,a_3\tau_2+0.14020267\,a_4\tau_1\\
&-0.026139479\,a_1\tau_1-0.04391202615\,a_1\tau_2-0.003021353873\,a_0\tau_1-0.008234564498\,a_0\tau_2\\
&-15.66079252\,a_1+9.7058251\,{a_1}^2-889.0249771\,a_4+40817.27237\,{a_4}^2+0.01722752345\,\tau_1\\
&+0.0002789882340\,{\tau_1}^2+0.03173491162\,\tau_2+0.0002789882340\,{\tau_2}^2-142.3508013\,a_2\\
&-4.352474682\,a_0-439.3729764\,a_3+800.1309000\,{a_2}^2+8221.304447\,{a_3}^2\\
&+0.7561983766\,{a_0}^2+6.333381647 \qquad (3.65)
\end{aligned}
$$

Evaluating the partial derivative of equation (3.60) with respect to the variables, we have

$$
\begin{aligned}
y(x) \;=\;& 8.99512915\,a_2+6.5417221\,a_4+5.352654208\,a_1+4.4817476\,a_3-0.003021353873\,\tau_1\\
&-0.008234564498\,\tau_2+1.512396753\,a_0-4.352474682=0 \qquad (3.66)
\end{aligned}
$$

$$
\begin{aligned}
y(x) \;=\;& 3.352654208\,a_2+1.5417221\,a_4+0.003021353873\,a_1+4.4817476\,a_3-6.99512915\,\tau_1\\
&-0.008234564498\,\tau_2+1.512396753\,a_0-2.352474682=0 \qquad (3.67)
\end{aligned}
$$

$$
\begin{aligned}
y(x) \;=\;& 8.1232652915\,a_2+3.589547221\,a_4+5.352654208\,a_1+7.48170576\,a_3-0.003021353873\,\tau_1\\
&-0.008234564498\,\tau_2+3.512396753\,a_0-7.352474682=0 \qquad (3.68)
\end{aligned}
$$

$$
\begin{aligned}
y(x) \;=\;& 5.352654208\,a_2+6.5417221\,a_4+4.4817476\,a_1+3.99512915\,a_3-0.003021353873\,\tau_1\\
&-0.008234564498\,\tau_2+1.912396753\,a_0-0.352474682=0 \qquad (3.69)
\end{aligned}
$$

$$
\begin{aligned}
y(x) \;=\;& 8.51291578\,a_2+5.546178221\,a_4+5.352654208\,a_1+4.4817476\,a_3-0.09703021353\,\tau_1\\
&-0.00213234564\,\tau_2+5.90512396753\,a_0-4.352474682=0 \qquad (3.70)
\end{aligned}
$$

Solving equations (3.66) to (3.69) using maple 18, we have

$$
\begin{aligned}
y(x) \;=\;& a_0=0.35416522,\, a_1=0.4237000130,\, a_2=1.676767090, \qquad (3.71)\\
& a_3=2.563731035\times 10^{-8},\, a_4=-3.54784170\times 10^{-9} \qquad (3.72)
\end{aligned}
$$

Substituting these values into equation (3.31), we have the approximate solution

$$y(x) = -1.0111 \times 10^{-6}\, x^4 + 0.000009624\, x^3 + 1.99999991\, x^2 + 1.000385766\, x + 1.421 \times 10^{-8} \quad (3.73)$$

For $\alpha = 2.5$. Solving example 3 as given from above, taking $\alpha = 2.5$ following the same procedure. We apply D^α for $\alpha = 2.5$ on equation (3.61), and after the necessary simplification and integrated from (0 to 1) as seen [17, 18], we have

$$\begin{aligned} y(x) \;=\; & 175.8530487\, a_1 a_2 + 48.99512915\, a_0 a_2 + 4961.882199\, a_2 a_3 + 536.8186141\, a_1 a_3 \\ & +1078.276113\, a_1 a_4 + 316.5417221\, a_0 a_4 + 35486.64754\, a_3 a_4 + 10080.68384\, a_2 a_4 \\ & +5.352654208\, a_0 a_1 + 154.4817476\, a_0 a_3 - 0.1045020835\, a_4 \tau_2 + 0.0004882294\, \tau_1 \tau_2 \quad (3.74) \end{aligned}$$

Evaluating the partial derivative of equation (3.72) with respect to the variables, we have

$$\begin{aligned} y(x) \;=\; & a_0 = 0.5419365, a_1 = 0.2637130, a_2 = 1.23507090, \\ & a_3 = 3.15035 \times 10^{-8}, a_4 = -1.184070 \times 10^{-9} \end{aligned}$$

Substituting these values into equation (2.13), we have the approximate solution

$$y(x) = -1.31 \times 10^{-6}\, x^4 + 0.000051234\, x^3 + 1.9999889\, x^2 + 0.980038216\, x + 1.421 \times 10^{-9} \quad (3.75)$$

as seen in [19, 20]. The results are shown in Table 3.3 and Figure 3.3 respectively.

Table 3.3: Error of Results for Example 3

x	Exact	$\alpha = 1.5$	Error	$\alpha = 2.5$	Error
0.0	0.0000000	0.000000142	1.4210e-11	0.00000008	8.0000e-10
0.1	0.1100000	0.120038724	1.0039e-10	0.11296763	2.9676e-10
0.2	0.2400000	0.252438085	1.2438e-10	0.28007738	4.0077e-09
0.3	0.3900000	0.419282585	2.9283e-09	0.48011612	9.0116e-09
0.4	0.5600000	0.720155044	1.6016e-09	0.61435307	5.4353e-09
0.5	0.7500000	0.838482284	8.8482e-09	1.00019419	2.5019e-09
0.6	0.9600000	1.092483720	1.3248e-08	1.32023350	3.6023e-08
0.7	1.1900000	1.377151690	1.8715e-07	1.68027340	4.9027e-08
0.8	1.4400000	1.693261268	2.5326e-06	2.08031358	6.4031e-08
0.9	1.710000	2.041568298	3.3157e-05	2.52035420	8.1035e-06
1.0	2.0000000	2.422809420	4.2281e-05	3.00039530	1.0004e-05

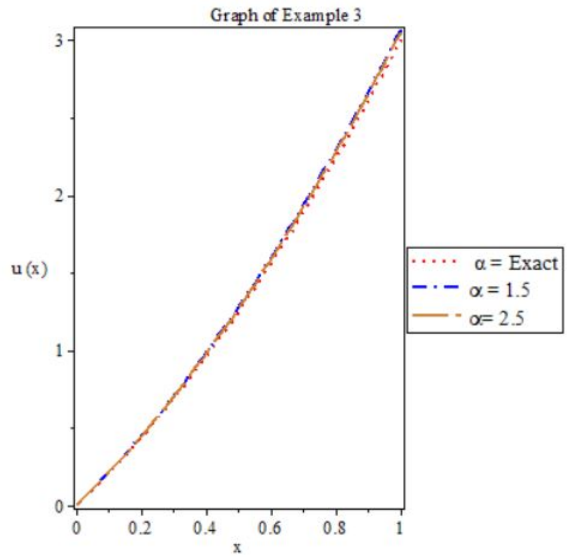

Figure 3.3: Graphical Representation of Error in Table 3

4. Conclusion

In this study, some classes of singular multi-order fractional differential equations were slightly perturbed and solved using the Akbari-Ganji method. Three examples were solved to demonstrate the proposed method (Akbari-Ganji method). From the tables and graphs it can be seen that the method performed very well with results rapidly converging to the exact solutions. Generally, the method is effective in solving the class of problems considered. Summarily, the exactness on the examples consider are tabulated for when $\alpha = 1.5$ and result obtained when $\alpha = 2.5$ for the error solutions.

Table 4.4: Result comparison for exact solution of examples 1 to 3 when $\alpha = 1.5$

x	Exact	$\alpha = 1.5$	Exact	$\alpha = 1.5$	Exact	$\alpha = 1.5$
0.0	0.0000000	-0.00400510	1.000000	0.000000	0.0000000	0.000000142
0.1	0.0110000	0.02207872	1.010050	1.010051	0.1100000	0.120038724
0.2	0.0480000	0.07447037	1.040810	1.040815	0.2400000	0.252438085
0.3	0.1170000	0.15103577	1.094174	1.094116	0.3900000	0.419282585
0.4	0.2240000	0.24992183	1.173510	1.173381	0.5600000	0.720155044
0.5	0.3750000	0.36955644	1.284025	1.282652	0.7500000	0.838482284
0.6	0.5760000	0.50864851	1.433329	1.432175	0.9600000	1.092483720
0.7	0.8330000	0.66618789	1.632316	1.630079	1.1900000	1.377151690
0.8	1.1520000	1.14144546	1.896480	1.897841	1.4400000	1.693261268
0.9	1.5390000	1.33973060	2.247907	2.235807	1.710000	2.041568298
1.0	2.0000000	1.86360322	2.718281	2.717813	2.0000000	2.422809420

Table 4.5: Result comparison for errors obtained for examples 1 to 3 when $\alpha = 1.5$ and $\alpha = 2.5$

Example 1		**Example 2**		**Example 3**	
$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 1.5$	$\alpha = 2.5$
4.051e-08	1.340e-07	1.6343e-07	3.0646e-08	1.4210e-11	8.0000e-10
1.279e-07	2.199e-07	1.5406e-07	1.5406e-07	1.0039e-10	2.9676e-10
3.470e-07	1.317e-06	2.7390e-07	2.7390e-06	1.2438e-10	4.0077e-09
3.136e-06	1.302e-06	1.2886e-06	1.2886e-06	2.9283e-09	9.0116e-09
5.922e-06	4.563e-06	3.5849e-05	3.5849e-05	1.6016e-09	5.4353e-09
1.156e-06	2.4912e-06	1.1521e-05	1.1521e-05	1.00019419	2.5019e-09
1.465e-05	3.317e-06	1.3014e-05	1.3014e-05	1.3248e-08	3.6023e-08
1.619e-05	1.740e-05	1.4008e-05	1.4008e-04	1.8715e-07	4.9027e-08
2.145e-05	1.653e-05	1.64472e-04	1.64472e-04	2.5326e-06	6.4031e-08
2.397e-05	2.184e-05	2.44011e-03	2.44016e-03	3.3157e-05	8.1035e-06
2.430e-05	2.021e-05	1.38143e-02	1.38147e-02	4.2281e-05	1.0004e-05

Recommendations

Considering the significant role of differential and fractional differential equations in modeling complex physical and engineering systems, further research should prioritize the development of more efficient analytical and numerical solution techniques. In addition, the potential applications of singular fractional differential equations in emerging areas such as quantum mechanics, biomedical engineering and financial mathematics should be extensively explored. Integrating these advanced mathematical techniques with appropriate computational and analytical skills will enhance the capacity to address complex dynamic systems and minimize associated analytical challenges.

Acknowledgment

Authors appreciate academic staffs of the Department of Mathematical Sciences of Osun State University, Osogbo, Nigeria. Reviewers of this manuscript are appreciated.

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Citation IEEE Format:
A. F. Adebisi and W. A. Oseni. *"Perturbed Akbari-Ganji Method for the Solution of Singular Multi-Order Fractional Differential Equations"*, *Jurnal Diferensial*, vol. 8(1), pp. 56-73, 2026.