J. DIFERENSIAL/E-ISSN:2775-9644
Vol.7(2),2025, 116–122
doi:https://doi.org/10.35508/jd.v7i2.20790

RESEARCH ARTICLE

The Metric Dimension of Theta Graph

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Received: 2 March 2025; Revised: 15 July 2025; Accepted: 30 July 2025; Published:1 November 2025.

Abstract:
Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices u and v in G, the shortest path distance between u and v is denoted by $d(u, v)$. If $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered set of vertices in the connected graph G , and $v \in V(G)$, then the representation of vertex v with respect to W , denoted as $r(v|W)$, is $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If $r(v|W)$ is distinct for each vertex $v \in V(G)$, then W is referred to as a resolving set for G. A resolving set with the smallest cardinality is called the minimum resolving set, and the cardinality of this set is the metric dimension of G, denoted by $\dim(G)$. This paper explores the metric dimension of the theta graphs.

Keywords: metric dimension, theta graph, basis, resolving set, graph, path graph, amalgamation

1. Introduction

$G = (V, E)$ be a connected graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. One of the topics in graph theory discussed in this research is the metric dimension. The concept of metric dimension was first introduced by Harary and Melter. In 2000, Chartrand et al. [1] also explored the topic of metric dimension. The distance between two vertices u and v is defined as the shortest path from u to v in G, denoted by $d(u, v)$. If an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ is given, then the representation of a vertex v with respect to W is $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If $r(v|W)$ is distinct for every vertex $v \in V(G)$, then W is called a resolving set. The minimum cardinality of a resolving set is called the metric dimension of G, denoted by $\dim(G)$.

With the advancement of science and technology, several results related to determining the metric dimension of certain graphs have been obtained. Some of the metric dimension results include the cycle graph [2], the fan graph [3], the wheel graph [4], the regular bipartite graph [5], the snowflake graph [6], and the subdivided-thorn graph [7]. Additionally, some results regarding the operation of graph addition have been obtained, such as by Suhud et al. [8], who determined the metric dimension of the windmill graph pattern $K_1 + mK_3$ for $m \geq 2$. Furthermore, Putra et al. [9] determined the metric dimension of the graph $W_n + C_n$ for $n \in \{3, 4\}$. In the same year, Utomo and Novian [10] obtained the metric dimension of the graph resulting from the amalgamation of n complete graphs K_m, for $n \geq 4$ and $m \geq 4$, denoted by $Amal\{nK_m|n \geq 4, m \geq 4\}$. Subsequently, Riyandho et al. [11]

obtained the metric dimension of the windmill graph pattern $K_1 + mK_4$ for $m \geq 2$, and corona graph by [12]. Ahmad et al. [13] examined the metric dimension of generalized Petersen graphs, offering insights into how the structure of these graphs affects the calculation of their metric dimension. This study highlighted that more complex graph structures require a specialized approach to determine their minimal resolving sets. Additionally, Shahida and Sunitha (2014) expanded this research by focusing on the metric dimension of the join of two graphs. They demonstrated that the interaction between two graph structures can significantly alter their metric dimension [14].

The study of metric dimensions has also been extended to more specific graphs, such as Grassmann graphs, as discussed by Bailey and Meagher [15]. In their research, they analyzed how the algebraic structure of Grassmann graphs influences their metric dimension, finding results that are relevant for theoretical computation applications. Fitriani and Cahyaningtyas continued this topic by exploring the metric dimension of dual antiprism graphs, enriching the understanding of how changes in graph structure impact their metric dimension values [16]. Recent research by Janan and Janan (2022) investigated the metric dimension of spider web graphs. They illustrated how certain network patterns affect the localization capabilities within the graph, which can have implications for studies in communication network design [17]. The metric dimension of amalgamation theta graph given by Wellyanti et al. [18].

Beyond the classical metric dimension, other variations have emerged to address more specific graph characteristics and requirements, further enriching the study of graph theory. One such variation is the edge metric dimension [19], which focuses on distinguishing edges rather than vertices, offering insights into the connectivity and traversal properties of a graph. Another extension is the k-metric dimension [20, 21], which generalizes the classic concept by requiring that each pair of vertices is distinguished by at least k vertices, providing a deeper understanding of the graph's structure and robustness. Moreover, the concept of the mixed metric dimension has also been explored. This variation combines aspects of both the vertex and edge metric dimensions [22], making it applicable to problems where both vertices and edges play critical roles in the graph's configuration. These variations of metric dimension have opened up new avenues for research, enabling more refined approaches to problems in areas like network security, navigation, and communication, where understanding the subtle structure of a graph is crucial.

Graph amalgamation is an operation on graphs. Let $\{G_1, G_2, \ldots, G_t\}$ for $i \in \{1, 2, \ldots, t\}$, $t \geq 2$, be a finite collection of nontrivial connected graphs, and let $v_{0,i}$ be a vertex from graph G_i called a terminal vertex. The amalgamated graph, denoted by $Amal\{G_i, v_{0,i}\}$, is a graph formed from $G_1, G_2, \ldots, G_t$ by identifying the terminal vertices from $\{G_1, G_2, \ldots, G_t\}$, such that $v_{0,1} = v_{0,2} = \cdots = v_{0,t}$. Simanjuntak et al. [23] found a theorem relating the metric dimension of a graph G to the amalgamation of the graph, as follows.

Theorem 1.1. *For $m \in \mathbb{N}$, $m \geq 2$, let $\{G_1, G_2, \ldots, G_m\}$ be a collection of arbitrary nontrivial connected graphs, and each G_j has a terminal vertex a_j, for $1 \leq j \leq m$. Let c be a new vertex resulting from the identification of all terminal vertices. If $G = Amal\{G_1, G_2, \ldots, G_m, c\}$, then*

$$\sum_{i=1}^{m} \dim(G_i) - m \leq \dim(G) \leq \sum_{i=1}^{m} \dim(G_i) + m - 1$$

The Theta graph, denoted as $\Theta(4, n)$, is a graph formed by performing a vertex amalgamation operation on the vertices $v_{k,1}$ for $1 \leq k \leq 4$ into a new vertex named a. Next, perform a vertex amalgamation operation on the vertices $v_{k,n}$ for $1 \leq k \leq 4$ into a new vertex named b. An illustration of the graphs $\Theta(4, n)$ is given in Figure 1.1.

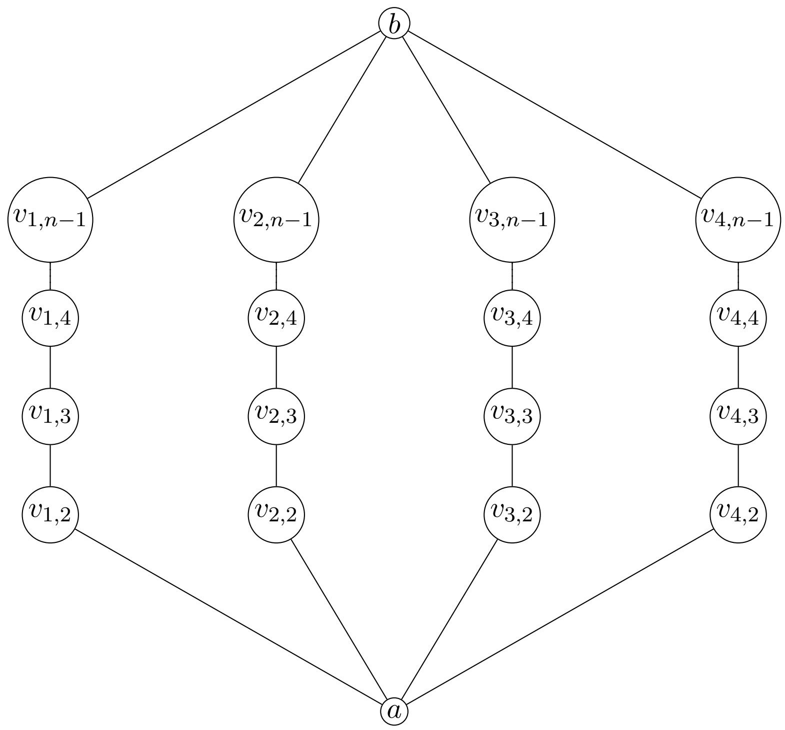

Figure 1.1: *Theta* (4,n) Graph

2. Research Method

The procedure for determining the metric dimension of the amalgamation graph involves several steps. Let H be amalgamated graph, to establish that $\dim(H) = k - 1$, we will set both upper and lower bounds for the metric dimension of graph H. The upper bound $\dim(H)$ is found by constructing a resolving set W with $|W| = k - 1$, ensuring that every vertex u in $V(H)$ has a unique representation. Conversely, the lower bound $\dim(H)$ is established by showing that for any resolving set W^* with $|W^*| = k-2$, there will always be at least two vertices that share the same representation.

3. Results and Discussion

3.1. Metric Dimension of Theta Graph

The following theorem provides the metric dimension of the Theta graph, denoted by $\Theta(4, n)$.

Theorem 3.1. *Let n be an integer, with $n \geq 3$. If $\Theta(4, n)$ is a Theta graph, then* $\dim(\Theta(4, n)) = 3$.

Proof. Given a graph $\Theta(4, n)$ with the vertex set $V(\Theta(4, n)) = \{a, b\} \cup \{v_{k,l} \mid 1 \leq k \leq 4, 2 \leq l \leq n - 1\}$. It will be shown that $\dim(\Theta(4, n)) = 3$. Without loss of generality, choose the set $W = \{v_{1,2}, v_{2,2}, v_{3,3}\} \in V(\Theta(4, n))$. It will be proven that $W = \{v_{1,2}, v_{2,2}, v_{3,3}\}$ is a resolving set. Consider the representation of all vertices $V(\Theta(4, n))$ as follows.

1. The representation of vertex a with respect to W is obtained as follows:

$$r(a \mid W) = (1, 1, 2).$$

2. The representation of vertex $v_{i,j}$, for each $1 \leq i \leq 4, 2 \leq j \leq n - 1$ with respect to W, is obtained as follows:

$$r(v_{1,2} \mid W) = (0, 2, 3),$$

$$r(v_{2,2} \mid W) = (2,0,3),$$
$$r(v_{3,2} \mid W) = (2,2,1),$$
$$r(v_{4,2} \mid W) = (2,2,3),$$
$$r(v_{1,3} \mid W) = (1,3,4),$$
$$r(v_{2,3} \mid W) = (3,1,4),$$
$$r(v_{3,3} \mid W) = (3,3,0),$$
$$\vdots$$
$$r(v_{1,n-2} \mid W) = (n-4, n-2, n-1),$$
$$r(v_{2,n-2} \mid W) = (n-2, n-4, n-1),$$
$$r(v_{3,n-2} \mid W) = (n-2, n-2, n-5),$$
$$r(v_{4,n-2} \mid W) = (n-2, n-2, n-1),$$
$$r(v_{1,n-1} \mid W) = (n-3, n-1, n-2),$$
$$r(v_{2,n-1} \mid W) = (n-1, n-3, n-2),$$
$$r(v_{3,n-1} \mid W) = (n-1, n-1, n-4).$$
$$r(v_{4,n-1} \mid W) = (n-1, n-1, n-2).$$

3. The representation of vertex b with respect to W is obtained as follows:

$$r(b \mid W) = (n-2, n-2, n-3).$$

Since each vertex in $\Theta(n)$ has a different representation, W is a resolving set, and consequently $\dim(\Theta(4,n)) \leq 3$. Next, it will be shown that for any W^* with $|W^*| = 2$, W^* is not a resolving set. That is, there will always be at least two vertices with the same representation with respect to W^*. Consider the following cases.
Let $W^* = \{\alpha, \beta\}$.

Case 1: If $W^* = \{\alpha, \beta\}$, then $\alpha \in \{a, b\}$, $\beta \in \{v_{i,j} \mid 1 \leq i \leq 4, 2 \leq j \leq n-1\}$. There will always exist a point $v_{r,s}$, with $1 \leq r \leq 4, r \neq i, 2 \leq s \leq n-1$, that has the same representation as point $v_{x,y}$, with $1 \leq x \leq 4, x \neq r, i, 2 \leq y \leq n-1, y = s$. Without loss of generality, let $W^* = \{a, v_{1,2}\}$, then the representations of points $v_{2,2}$ and $v_{3,2}$ with respect to W^* are as follows:

$r(v_{2,2}) = (1,2) = r(v_{3,2})$.

Case 2: If $W^* = \{\alpha, \beta\}$, then $\alpha \in \{v_{i,j} \mid 1 \leq i \leq 4, 2 \leq j \leq n-1\}$, $\beta \in \{v_{k,j} \mid 1 \leq k \leq 4, k \neq i, 2 \leq j \leq n-1\}$. Consider the following subcases:

Subcase 2.1: $n = 3$. Choose any point from the set of points $v_{i,j}$ and $v_{k,j}$, there will always be a point with the same representation on a and b.

Subcase 2.2: $n > 3$, n is even. 1. Let $W^* = \{v_{i,j}, v_{k,j}\}$, with $j \leq \frac{n}{2}$. There will always exist a point $v_{x,y}$, with $1 \leq x \leq 4, x \neq i, k, 2 \leq y \leq n-1$, that has the same representation as point b. 2. Let $W^* = \{v_{i,j}, v_{k,j}\}$, with $j > \frac{n}{2}$. There will always exist a point $v_{x,y}$, with $1 \leq x \leq 4, x \neq i, k, 2 \leq y \leq n-1$, that has the same representation as point a.

Subcase 2.3: $n > 3$, n is odd. 1. Let $W^* = \{v_{i,j}, v_{k,j}\}$, with $j < \lceil \frac{n}{2} \rceil$. There will always exist a point $v_{x,y}$, with $1 \leq x \leq 4, x \neq i, k, 2 \leq y \leq n-1$, that has the same representation as point b. 2. Let $W^* = \{v_{i,j}, v_{k,j}\}$, with $j = \lceil \frac{n}{2} \rceil$. There will always exist a point with the same representation as points a and b. 3. Let $W^* = \{v_{i,j}, v_{k,j}\}$, with $j > \lceil \frac{n}{2} \rceil$. There will always exist a point $v_{x,y}$, with $1 \leq x \leq 4, x \neq i, k, 2 \leq y \leq n-1$, that has the same representation as point a.

Case 3: If $W^* = \{\alpha, \beta\}$, then $\alpha \in \{v_{i,j} \mid 1 \leq i \leq, 2 \leq j \leq n-1\}$, $\beta \in \{v_{x,y} \mid 1 \leq x \leq 4, x \neq i, 2 \leq y \leq n-1, y \neq j\}$. Without loss of generality, let $W^* = \{v_{1,2}, v_{2,3}\}$, then the representations of points $\{v_{1,n-1}, v_{3,n-3}\}$ with respect to W^* are as follows:
$r(v_{1,n-1} \mid W) = (n-3, n-3) = r(v_{3,n-3} \mid W)$.

Case 4: If $W^* = \{\alpha, \beta\}$, then $\alpha \in \{v_{i,j} \mid 1 \leq i \leq 3, 2 \leq j \leq n-1\}$, $\beta \in \{v_{i,k} \mid 1 \leq i \leq 4, 2 \leq k \leq n-1, k \neq j\}$. There will always exist a point $v_{r,s}$, with $1 \leq r \leq 4, r \neq i, 2 \leq s \leq n-1$, that has the same representation as point $v_{x,y}$, with $1 \leq x \leq 3, x \neq r, i, 2 \leq y \leq n-1, y = s$. Without loss of generality, let $W^* = \{v_{1,2}, v_{1,3}\}$, then the representations of points $\{v_{2,2}, v_{3,2}\}$ with respect to W^* are:
$r(v_{2,2} \mid W) = (2, 3) = r(v_{3,2} \mid W)$.

From the four cases above, since there are points with the same representation with respect to W^*, it is proven that W^* is not a resolving set for the Theta graph. Consequently, $\dim(\Theta(4, n)) \geq 3$. Therefore, $\dim(\Theta(4, n)) = 3$. □

3.2. Time Complexity

The time complexity for determining the metric dimension of a theta graph involves analyzing how efficiently one can identify a resolving set for the graph. A theta graph, typically consisting of multiple paths with standard endpoints, may allow more efficient determination of its metric dimension than general graphs due to its structured nature. The complexity depends on the number of vertices and edges, and specialized algorithms can sometimes achieve polynomial time solutions for certain classes of graphs, like trees or specific types of theta graphs. Here is the running time from the program of metric dimensin of theta graph $\Theta(4, n)$.

Table 3.1: The Metric Dimension and Running Time

n	$\dim(\Theta(4, n))$	Running Time
5	3	3956.31 ms
10	3	8400.71 ms
15	3	18842.26 ms
20	3	54948.52 ms
25	3	129569.02 ms
30	3	293652.51 ms

4. Conclusion

For a theta graph $\Theta(4, n)$, where,$n \geq 3$, the metric dimension is consistently 3. This result implies that regardless of how long the paths are (as long as there are at least three paths), only three vertices are needed to form a resolving set. This is due to the structure of the graph, where the three chosen vertices can uniquely determine the distances to any other vertex, effectively distinguishing all vertices based on their distance vectors. The problem is known to be NP-hard, meaning that no known polynomial-time algorithm can solve it for all graph instances. However, for future research, consider exploring the relationship between the metric dimension of more complex theta graphs (e.g., $\Theta(k, n)$ for $k > 4$) and how the number of paths affects the resolving set size.

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Citation IEEE Format:
D. Rahmadi and W. F. Albar. *"The Metric Dimension of Theta Graph"*, *Jurnal Diferensial*, vol. 7(2), pp. 116-122, 2025.