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RESEARCH ARTICLE

Nonlocal Metric Dimension of Windmill Graph

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Abstract:
Let $G = (V(G), E(G))$ be a simple and connected graph. The distance between two vertices u and v in G, is the length of a shortest path from u to v, denoted by $d(u, v)$. Suppose $S = \{s_1, s_2, ...s_k\}$ is an ordered subset of vertices of G, then the metric representation of a vertex $u \in V(G)$ with respect to S, denoted by $r(u|S)$, is the $k-$vector $(d(u, s_1), d(u, s_2), ..., d(u, s_k))$. If every two nonadjacent vertices of G have distinct metric representations with respect to S, then the set S is called a nonlocal resolving set for G. A nonlocal resolving set with minimum cardinality is called a nonlocal metric basis. The nonlocal metric dimension of G is the cardinality of the nonlocal metric basis of G and is denoted by $nldim(G)$. In this paper, we obtained nonlocal metric dimension of windmill graph.

Keywords: nonlocal metric dimension, nonlocal resolving set, windmill graph.

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1. Introduction

One topic in graph theory that has been extensively studied by researchers is metric dimension. The concept of metric dimensions on a graph was first introduced by Slater [1] and Harary and Melter [2]. Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u, v \in V(G)$, the distance $d(u, v)$ is the length of the shortest path between u and v. Given $S = \{s_1, s_2, ..., s_k\} \subseteq V(G)$, the metric representation of a vertex u with respect to S is defined as

$$r(u|S) = (d(u, s_1), d(u, s_2), ..., d(u, s_k)).$$

If every pair of distinct vertices of G has different metric representations, then S is called a resolving set. A resolving set of minimum cardinality is called a metric basis, and its cardinality is referred to as the metric dimension of G, denoted by $dim(G)$. The metric dimension has wide-ranging applications, such as robot navigation [3], chemical compound classification [4], coin weighing problems [5], and the placement of fire sensors in buildings [6].

Over the years, the study of metric dimension has expanded into various variants such as the strong metric dimension [7], local metric dimension [8], the k-metric dimension [9], edge metric dimension [10], and nonlocal metric dimension [11]. In the local metric dimension, the focus is on resolving every adjacent pair of vertices, while in the edge metric dimension, the objects being resolved are the edges of the graph. More recently, Klavžar and Kuziak [11] introduced the nonlocal metric dimension, where the requirement is to resolve every nonadjacent pair of vertices.

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Formally, in a simple connected graph G, a set $S \subseteq V(G)$ is called a nonlocal resolving set if every two nonadjacent vertices $u, v \in V(G)$ have distinct metric representations with respect to S, that is

$$r(u|S) \neq r(v|S).$$

The nonlocal resolving set with minimum cardinality is called a nonlocal metric basis, and its cardinality is the nonlocal metric dimension of G, denoted by $nldim(G)$.

Recent studies have started to explore this new variant. Klavžar and Kuziak [11] introduced the concept of nonlocal metric dimension and provided characterizations for graphs with $nldim(G) = 1$ and $nldim(G) = V(G) - 2$, as well as results for block graphs, corona products, and wheels. More recently, Xiong and Deng [12] investigated nonlocal metric dimension in several network models, including hexagonal ladder, radix triangular mesh, Sierpiński, and small-world networks. These works demonstrate that although the concept is new, it has begun to attract attention in both theoretical and applied contexts.

This motivates further exploration of the nonlocal metric dimension in different graph classes. In this paper, we focus on determining the nonlocal metric dimension of the windmill graph. It should be noted that the metric dimension and edge metric dimension of windmill graph have been investigated in [13], while the study on its local metric dimension was presented in [14].

2. Methodology

The method used in this study is a literature review, which involves collecting references related to research on nonlocal metric dimensions and relevant topics obtained from articles, proceedings, and books. The steps taken to determine the nonlocal metric dimension of windmill graph are as follows.

1. Construct the windmill graph.
2. Determine the distances between vertices in windmill graph.
3. Select nonlocal resolving sets of windmill graph.
4. Identify the nonlocal resolving set with minimum cardinality.
5. Establish the general (explicit) formula for nonlocal metric dimension of windmill graph.
6. Develop a theorem and provide a proof of its validity.

To show that nonlocal metric dimension of a graph G is $nldim(G) = k$, it must be proven that $nldim(G) \leq k$ and $nldim(G) \geq k$. The upper bound can be shown by demonstrating that there exists a nonlocal resolving set $S \subseteq V(G)$ with $|S| = k$. Meanwhile, the lower bound can be established by showing that for every possible set S' with $|S'| \leq k - 1$ is not a nonlocal resolving set of G, that is, there exist two nonadjacent vertices with same metric representation.

3. Results and Discussions

Definition 3.1. [15] Windmill graph W_n^m is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

Figure 3.1 illustrates the windmill graph W_5^3. In general, the windmill graph W_n^m has order $m(n-1)+1$ and size $\frac{mn(n-1)}{2}$.

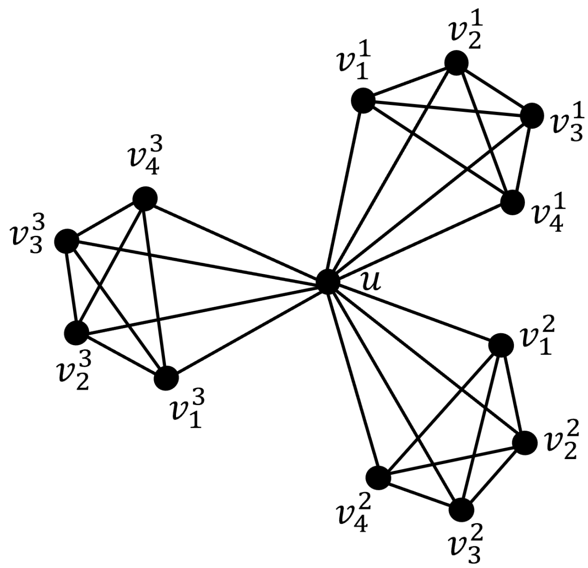

Figure 3.1: Windmill graph W_5^3

In windmill graph W_n^m, the distance between each pair of different vertices is given in Table 3.1.

Table 3.1: The distance between each pair of different vertices in W_n^m

No	Distance
1	$d(u, v_j^i) = 1$ with $i = 1, 2, ..., m$ and $j = 1, 2, ..., n-1$
2	$d(v_j^i, v_k^i) = 1$ with $i = 1, 2, ..., m$ and $j, k = 1, 2, ..., n-1$ $(j \neq k)$
3	$d(v_k^i, v_l^j) = 2$ with $i, j = 1, 2, ..., m$ and $k, l = 1, 2, ..., n-1$ $(i \neq j)$

From Table 3.1, we observe that two nonadjacent vertices are v_k^i and v_l^j with $i, j = 1, 2, \ldots, m$; $k, l = 1, 2, \ldots, n-1$; and $i \neq j$. Furthermore, lemmas and a theorem are given regarding nonlocal metric dimension of the windmill graph W_n^m, which is the result of this research. To simplify the proof, the set $F_i (i = 1, 2, \ldots, m)$ is defined as follows.

$$F_i = \{v_1^i, v_2^i, \ldots, v_{n-1}^i\}$$

Lemma 3.1. *A set S is a nonlocal resolving set of W_n^m with $m \geq 2$ and $n \geq 3$ if and only if for every pair F_i, F_j with $i \neq j$, $S \cap (F_i \cup F_j) \neq \emptyset$.*

Proof. Assume that $S \cap (F_i \cup F_j) = \emptyset$ for some $i \neq j$. Choose $v_1^i \in F_i$ and $v_1^j \in F_j$. Since $i \neq j$, we have $d(v_1^i, v_1^j) = 2$, which means that v_1^i and v_1^j are nonadjacent vertices and, therefore, should be resolved nonlocally. Note that if $u \in S$, then $d(u, v_1^i) = d(u, v_1^j) = 1$. Moreover, for every $s \in S \setminus \{u\}$, since $s \notin F_i \cup F_j$, the shortest path from s to either v_1^i or v_1^j necessarily passes through u. Thus, $d(s, v_1^i) = d(s, v_1^j) = 2$. Consequently, $r(v_1^i|S) = r(v_1^j|S)$, contradicting the fact that S is a nonlocal resolving set. Hence, $S \cap (F_i \cup F_j) \neq \emptyset$ for every $i \neq j$.
Let $x \in F_i$ and $y \in F_j$ be two nonadjacent vertices with $i \neq j$. Since $S \cap (F_i \cup F_j) \neq \emptyset$, there exists $s \in S \cap (F_i \cup F_j)$.

- If $s \in F_i$, then $d(s, x) \in \{0, 1\}$, where $d(s, x) = 0$ when $s = x$ and $d(s, x) = 1$ when $s \neq x$. In either case, we have $d(s, y) = 2$.
- If $s \in F_j$, then $d(s, y) \in \{0, 1\}$, where $d(s, y) = 0$ when $s = y$ and $d(s, y) = 1$ when $s \neq y$. In either case, we have $d(s, x) = 2$.

In both cases, it follows that $d(s,x) \neq d(s,y)$, and thus x and y have different metric representations with respect to S. Therefore, S is a nonlocal resolving set. $\square$

Lemma 3.2. *If S is a nonlocal resolving set of W_n^m with $m \geq 2$ and $n \geq 3$, then $|S| \geq m-1$.*

Proof. Assume to the contrary that $|S| \leq m-2$, then there exist at least two sets F_i and F_j with $i \neq j$ such that $S \cap (F_i \cup F_j) = \emptyset$. By Lemma 3.1, this implies that S is not a nonlocal resolving set, a contradiction. Hence, $|S| \geq m-1$. $\square$

Theorem 3.1. *If $m \geq 2$ and $n \geq 3$, then $nldim(W_n^m) = m-1$.*

Proof. Let $S = \{v_1^1, v_1^2, ..., v_1^{m-1}\}$ with $m \geq 2$ (see Figure 3.2).

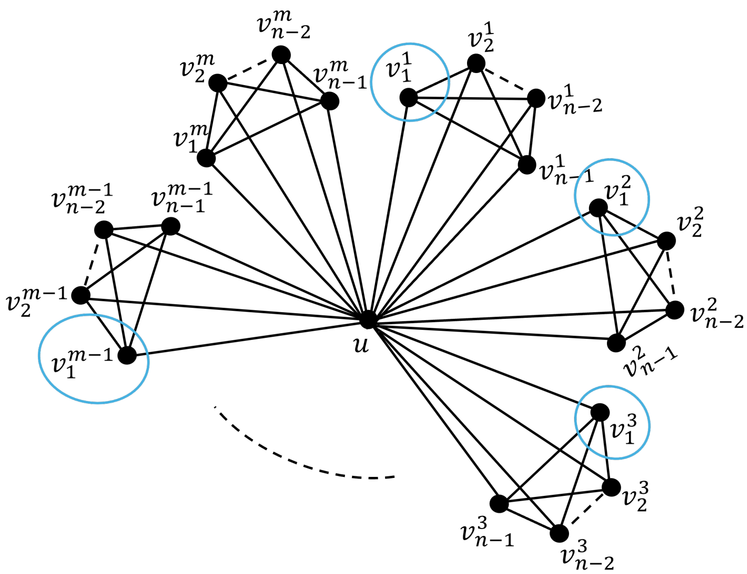

Figure 3.2: Windmill graph W_n^m

We show that S is a nonlocal resolving set of the windmill graph W_n^m with $m \geq 2$ and $n \geq 3$. Consider the following metric representations:

$$r(u \mid S) = (1,1,1,1,\ldots,1,1)$$
$$r(v_k^1 \mid S) = (1,2,2,2,\ldots,2,2,2)$$
$$r(v_k^2 \mid S) = (2,1,2,2,\ldots,2,2,2)$$
$$r(v_k^3 \mid S) = (2,2,1,2,\ldots,2,2,2)$$
$$\vdots$$
$$r(v_k^{m-1} \mid S) = (2,2,2,2,\ldots,2,1,2)$$
$$r(v_k^m \mid S) = (2,2,2,2,\ldots,2,2,2)$$

with $k = 1,2,\ldots,n-1$. It follows that for every two nonadjacent vertices v_k^i dan v_l^j with $i,j = 1,2,\ldots,m$; $k,l = 1,2,...,n-1$, and $i \neq j$, their metric representations are distinct. Hence, S is a nonlocal resolving set and we obtain

$$|S| \leq m-1. \tag{3.1}$$

From Lemma 3.2 and inequality 3.1, it follows that S is a nonlocal metric basis. Consequently, the nonlocal metric dimension of the windmill graph is $m-1$ □

Our result on nonlocal metric dimension of windmill graph W_n^m can be compared with results for other variants. For instance, Singh et al. [13] computed metric dimension and edge metric dimension of windmill graph, showing that $dim(W_n^m) = m(n-2)$ and $edim(W_n^m) = m(n-1)-1$. Cynthia and Fancy [14] showed that the local metric dimension of windmill graphs coincides with the metric dimension, that is $dim(W_n^m) = ldim(W_n^m) = m(n-2)$. This demonstrates that nonlocal metric dimension provides a complementary perspective not captured by metric dimension, local metric dimension, or edge metric dimension. This study is limited to windmill graph, and the results cannot be directly extended to other graph classes. Special cases occur when $n=3$ and $m \geq 2$, windmill graph is isomorphic to friendship graph F_m. Therefore, nonlocal metric dimension of friendship graph is $nldim(F_m) = m-1$. Further research may explore the nonlocal metric dimension in other classes of graphs.

4. Conclusion

This study examined the nonlocal metric dimension of windmill graph. From the proven lemmas and theorem, it can be concluded that for $m \geq 2$ and $n \geq 3$, the value of the nonlocal metric dimension of the windmill graph W_n^m is $m-1$. Furthermore, research related to nonlocal metric dimensions can be examined for other classes of graphs.

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