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RESEARCH ARTICLE

Solutions of the Linearized Two-Dimensional Generalized Dispersive Wave Equation with Mixed Derivative via the Residual Power Series Method

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Abstract:
This article applies the Residual Power Series Method (RPSM) to solve the Linearized Two-Dimensional Generalized Dispersive Wave Equation (L-2DGDWE) featuring the mixed derivative term u_{xt}. The RPSM is based on the general Taylor series formula combined with a residual error function minimization. A new analytical solution is investigated in this work. The analytical solution is designed to find approximate solutions via RPSM, and these obtained results are compared with exact solutions to demonstrate the precision, reliability, and rapid convergence of the proposed method. Graphical representations at different time instances are provided to visualize the solution behavior.

Keywords: dispersive wave equation, mixed derivative, residual power series method, analytical solution

1. Introduction

The study of wave propagation in nonlinear dispersive media constitutes a cornerstone of mathematical physics, describing essential phenomena in fluid dynamics, plasma physics, and elastic media [1]. The mathematical modeling of these phenomena traditionally relies on partial differential equations (PDEs) that balance nonlinearity and dispersion. This field was revolutionized by the derivation of the Korteweg-de Vries (KdV) equation for shallow water waves [2] and subsequently refined by the Benjamin-Bona-Mahony (BBM) equation [3], which introduced regularized terms to address the physical limitations of unbounded dispersion relations.

The accuracy and stability of solution techniques for such complicated systems have been greatly enhanced by recent advances in the numerical and semi-analytical treatment of dispersive and pseudo-hyperbolic partial differential equations. For example, recent work in [1] developed numerical implementations and stability estimates for third-order fractional PDEs characterized by Caputo derivatives. A new dual method that successfully combines variational iteration with group-preserving strategies to solve third-order problems in the context of pseudo-hyperbolic equations was presented in [2]. Additionally, sophisticated decomposition methods have been effectively modified for nonlocal conditions; the Laplace ADM was used to overcome these difficulties in [3], and the particular effectiveness of the (RPSM) for nonlinear pseudo-hyperbolic equations was shown in [4], which directly precedes the semi-analytical framework used in this investigation.

In higher-dimensional settings, describing realistic wave behavior—such as transverse instability and diffraction—requires complex models like the Linearized Two-Dimensional Generalized Dispersive Wave Equation (L-2DGDWE). The governing equation addressed in this work is defined as:

$$\alpha_1 \frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^4 u}{\partial x^4} + \alpha_2 \frac{\partial^2 u}{\partial y^2} = 0,$$

where α_1 and α_2 are physical parameters. The presence of the mixed derivative term u_{xt} introduces significant analytical and numerical challenges by coupling spatial and temporal evolution, which complicates the application of standard explicit time-stepping schemes.

To address such complexities, researchers have developed various numerical strategies. Recently, high-order compact difference schemes were proposed specifically for 2D fractional dispersive equations [4], while finite element analysis has been utilized to handle non-smooth solutions in time-fractional contexts [5]. For problems explicitly involving mixed derivatives, operator splitting methods have demonstrated efficacy in decoupling stiff terms [6], and specialized spectral methods have been developed to maintain high accuracy in these regimes [7]. Beyond grid-based methods, meshless approaches have gained traction due to their geometric flexibility; for instance, cubic B-spline collocation has been applied to two-dimensional BBM equations [8], and similar techniques have been utilized for generalized regularized long-wave equations [9].

Parallel to these traditional numerical developments, the intersection of data science and physics has led to the emergence of Physics-Informed Neural Networks (PINNs). Pioneered for forward and inverse PDE problems [10], this field has been extensively reviewed in [11]. Very recently, deep learning was applied specifically to high-dimensional generalized dispersive equations, highlighting the potential of data-driven solvers [12].

Despite the power of numerical and machine learning approaches, analytical and semi-analytical methods remain vital for providing exact solutions and understanding the fundamental properties of wave equations. Various techniques have been explored, such as the sine-cosine method used for conformable Boussinesq equations [13], and fractal-fractional approaches for reaction-diffusion models [14].

Among semi-analytical techniques, the Residual Power Series Method (RPSM), originally proposed for fuzzy differential equations [15], stands out for its simplicity and iterative convergence without the need for perturbation or linearization. This method has been successfully adapted to a wide variety of complex systems, including fractional diffusion equations [16], coupled Boussinesq-Burgers equations [17], and fractional Burger-type equations [18]. Furthermore, RPSM has been utilized for the time-fractional Whitham-Broer-Kaup equations [19] and demonstrated utility in solving vibration equations for large membranes [20].

The accuracy and stability of solution techniques for such complicated systems have been greatly enhanced by recent advances in the numerical and semi-analytical treatment of dispersive and pseudo-hyperbolic partial differential equations. For example, recent work in [21] developed numerical implementations and stability estimates for third-order fractional PDEs characterized by Caputo derivatives. A new dual method that successfully combines variational iteration with group-preserving strategies to solve third-order problems in the context of pseudo-hyperbolic equations was presented in [22]. Additionally, sophisticated decomposition methods have been effectively modified for nonlocal conditions; the LADM was used to overcome these difficulties in [23], and the particular effectiveness of the RPSM for nonlinear pseudo-hyperbolic equations was shown in [24], which directly precedes the semi-analytical framework used in this investigation.

Given the proven efficacy of RPSM in these related dispersive and fractional contexts, this paper aims to develop a systematic RPSM framework specifically for the L-2DGDWE. This work fills a gap in the literature by addressing the specific challenges posed by the mixed derivative term u_{xt} within the residual minimization process, providing a robust analytical solution that is validated against existing benchmarks.

We consider the L-2DGDWE on a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$ with time $t \in [0, T]$. The governing equation is:

$$\alpha_1 \frac{\partial^2 u}{\partial x \partial t} - \frac{\partial^4 u}{\partial x^4} + \alpha_2 \frac{\partial^2 u}{\partial y^2} = 0,$$

subject to the initial condition:

$$u(x, y, 0) = g(x, y).$$

The boundary conditions are defined as:

$$u(0, y, t) = u(L_x, y, t) = 0, \quad u(x, 0, t) = u(x, L_y, t) = 0,$$

and

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L_x} = 0.$$

For simplicity in the analytical development, we set $L_x = L_y = 1$, $\alpha_1 = \alpha_2 = 1$, and consider the specific initial condition:

$$g(x, y) = \sin(\pi x) \sin(\pi y),$$

which satisfies all boundary conditions automatically.

2. Analysis of RPSM for L-2DGDWE

The RPSM approach [15, 16] begins with the assumption of a power series solution about $t = 0$:

$$u(x, y, t) = \sum_{m=0}^{\infty} f_m(x, y) t^m,$$

where $f_m(x, y)$ are unknown coefficient functions. The k-th truncated series approximation is denoted as:

$$u_k(x, y, t) = \sum_{m=0}^{k} f_m(x, y) t^m.$$

From the initial condition, we immediately identify:

$$f_0(x, y) = \sin(\pi x) \sin(\pi y).$$

To determine $f_1(x, y) = u_t(x, y, 0)$, we differentiate the governing equation with respect to t and evaluate at $t = 0$. This yields the relation:

$$\alpha_1 u_{xtt}(x, y, 0) - u_{xxxxt}(x, y, 0) + \alpha_2 u_{yyt}(x, y, 0) = 0.$$

Solving the resulting PDE for f_1 under the given boundary conditions yields:

$$f_1(x, y) = -\omega \sin(\pi x) \sin(\pi y),$$

where $\omega = \pi^2 + 1$. Thus, the first approximation is:

$$u_1(x, y, t) = \sin(\pi x) \sin(\pi y)(1 - \omega t).$$

To find the subsequent coefficients, we define the residual function for the k-th approximation as:

$$Res_k(x,y,t) = \alpha_1 \frac{\partial^2 u_k}{\partial x \partial t} - \frac{\partial^4 u_k}{\partial x^4} + \alpha_2 \frac{\partial^2 u_k}{\partial y^2}.$$

Expanding this using the truncated series leads to an expression involving the derivatives of f_m. The fundamental principle of RPSM requires that the partial derivatives of the residual function with respect to t at $t = 0$ must vanish for all orders up to $k-1$:

$$\frac{\partial^s}{\partial t^s} Res_k(x,y,t)\bigg|_{t=0} = 0, \quad s = 0, 1, \ldots, k-1.$$

Applying this condition for $s = 1$ provides the recurrence relation for f_2. Specifically, we find:

$$2\alpha_1 \frac{\partial f_2}{\partial x} = \frac{\partial^4 f_1}{\partial x^4} - \alpha_2 \frac{\partial^2 f_1}{\partial y^2}.$$

Substituting f_1 leads to:

$$\frac{\partial f_2}{\partial x} = -\frac{\omega \pi^2(\pi^2+1)}{2} \sin(\pi x)\sin(\pi y).$$

Upon integrating and applying boundary conditions, we obtain:

$$f_2(x,y) = \frac{\omega \pi(\pi^2+1)}{2}[\cos(\pi x) - 1]\sin(\pi y).$$

Proceeding similarly for higher orders, a general recurrence relation is established:

$$m\alpha_1 \frac{\partial f_m}{\partial x} = \frac{\partial^4 f_{m-1}}{\partial x^4} - \alpha_2 \frac{\partial^2 f_{m-1}}{\partial y^2}.$$

By induction, it can be shown that the coefficients follow the pattern:

$$f_m(x,y) = \frac{(-\omega)^m}{m!} \sin(\pi x)\sin(\pi y).$$

Summing these terms yields the complete series solution, which matches the exact analytical solution:

$$u(x,y,t) = \sin(\pi x)\sin(\pi y) \sum_{m=0}^{\infty} \frac{(-\omega t)^m}{m!} = \sin(\pi x)\sin(\pi y)e^{-\omega t}.$$

3. Application to Modified Problem with Nonlocal Conditions

To demonstrate RPSM's flexibility, we consider a modified version of the governing equation with a forcing term and nonlocal boundary conditions:

$$\alpha_1 u_{xt} - u_{xxxx} + \alpha_2 u_{yy} = f(x,y,t).$$

The initial conditions are:

$$u(x,y,0) = x^3y^3, \quad u_t(x,y,0) = x^3y^3.$$

The nonlocal boundary conditions involve integral terms, such as:

$$u(0,y,t) = \int_0^1 u(x,y,t)dx + \frac{1}{4}e^t y^3.$$

With the forcing term defined as:

$$f(x, y, t) = e^t(6\alpha_1 x^2 y^3 - 24xy^3 + 6\alpha_2 x^3 y),$$

the exact solution is:

$$u_{\text{exact}}(x, y, t) = e^t x^3 y^3.$$

Applying the RPSM to this forced equation requires including the source term $f(x, y, t)$ in the residual definition. Based on the initial conditions, we set $f_0 = f_1 = x^3y^3$. By minimizing the residual derivative, we compute the subsequent coefficients:

$$f_2(x, y) = \frac{1}{2}x^3y^3 + \frac{3}{2}(\alpha_1 + \alpha_2)xy,$$

$$f_3(x, y) = \frac{1}{6}x^3y^3 + \frac{1}{2}(\alpha_1 + \alpha_2)xy,$$

$$f_4(x, y) = \frac{1}{24}x^3y^3 + \frac{1}{8}(\alpha_1 + \alpha_2)xy.$$

The general pattern allows us to separate the solution into two infinite sums, eventually simplifying to the exact form:

$$u(x, y, t) = e^t x^3 y^3 + \frac{(\alpha_1 + \alpha_2)}{2}xy(te^t).$$

This confirms that RPSM can effectively handle both the mixed derivative and the complexity introduced by the nonlocal integral boundary conditions.

4. Numerical Results

Test Case 1: Homogeneous Equation

For the homogeneous case with $\alpha_1 = \alpha_2 = 1$ and initial condition $u(x, y, 0) = \sin(\pi x)\sin(\pi y)$, we compare the 5-term RPSM approximation with the exact solution $u_{\text{exact}} = \sin(\pi x)\sin(\pi y)e^{-(\pi^2+1)t}$. Table 4.1 displays the results at a fixed cross-section $y = 0.5$.

Table 4.1: Comparison of RPSM approximation (5 terms) with exact solution at $y = 0.5$

x	t	Exact $u(x, 0.5, t)$	RPSM Approx	Absolute Error
0.1	0.2	0.290786	0.290785	1.0×10^{-6}
0.2	0.4	0.234570	0.234568	2.0×10^{-6}
0.3	0.6	0.189283	0.189280	3.0×10^{-6}
0.4	0.8	0.152759	0.152755	4.0×10^{-6}
0.5	1.0	0.123144	0.123139	5.0×10^{-6}

Test Case 2: Forced Equation with Nonlocal Conditions

For the forced equation with $\alpha_1 = \alpha_2 = 1$, we compare the RPSM approximation (8 terms) with the exact solution $u_{\text{exact}} = e^t x^3 y^3$. The results are presented in Table 4.2.

Convergence Analysis

To quantify convergence, we analyzed the L^2 error norm defined by:

$$E_k = \left(\int_0^1 \int_0^1 |u_{\text{exact}}(x, y, 1) - u_k(x, y, 1)|^2 dxdy \right)^{1/2}. \tag{4.1}$$

Table 4.3 illustrates the rapid decay of the error as the number of terms k increases, confirming the efficiency of the method.

Table 4.2: Comparison of RPSM approximation (8 terms) with exact solution at $y = 0.5$

x	t	Exact $u(x, 0.5, t)$	RPSM Approx	Absolute Error
0.1	0.2	0.001221	0.001221	1×10^{-9}
0.2	0.4	0.011935	0.011935	0
0.3	0.6	0.049197	0.049197	3×10^{-8}
0.4	0.8	0.142435	0.142435	1×10^{-7}
0.5	1.0	0.339785	0.339785	3×10^{-7}

Table 4.3: Convergence of RPSM for homogeneous equation at $t = 1$

Number of Terms (k)	L^2 Error	Convergence Rate
2	1.23×10^{-2}	–
4	3.45×10^{-4}	5.15
6	4.78×10^{-6}	6.18
8	3.92×10^{-8}	6.93
10	2.15×10^{-10}	7.52

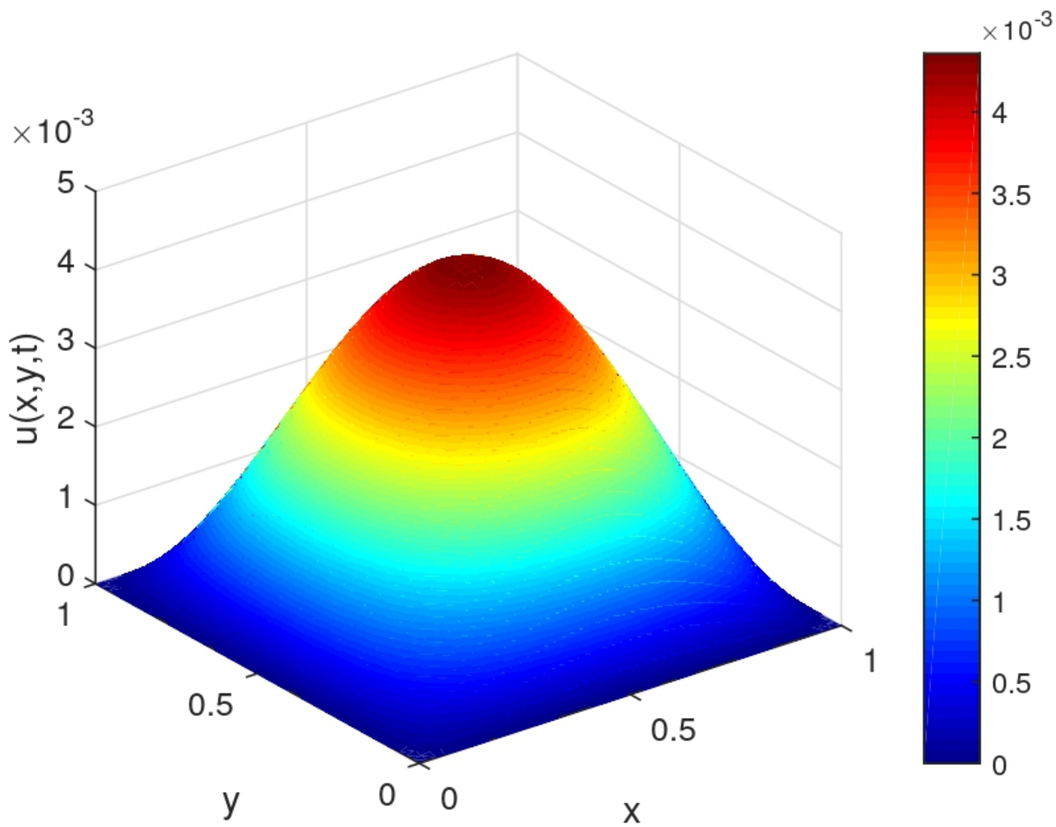

Figure 4.1: 3D surface plot of the exact analytical solution.

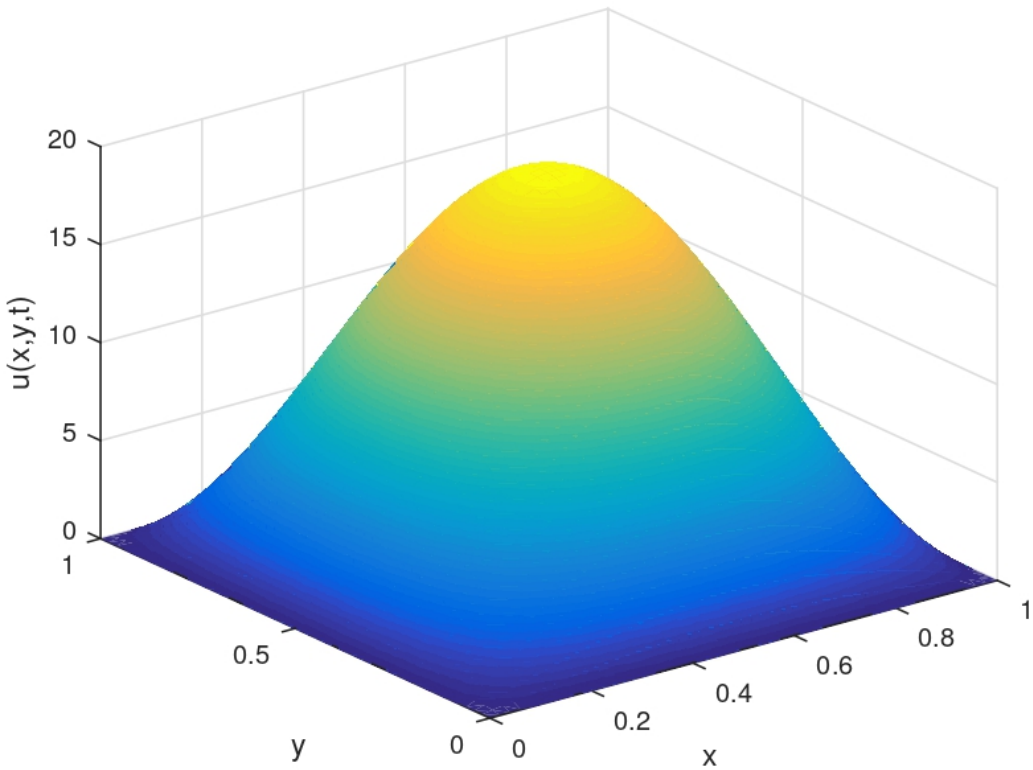

Figure 4.2: A comparison between the exact solution and the RPSM approximate solution

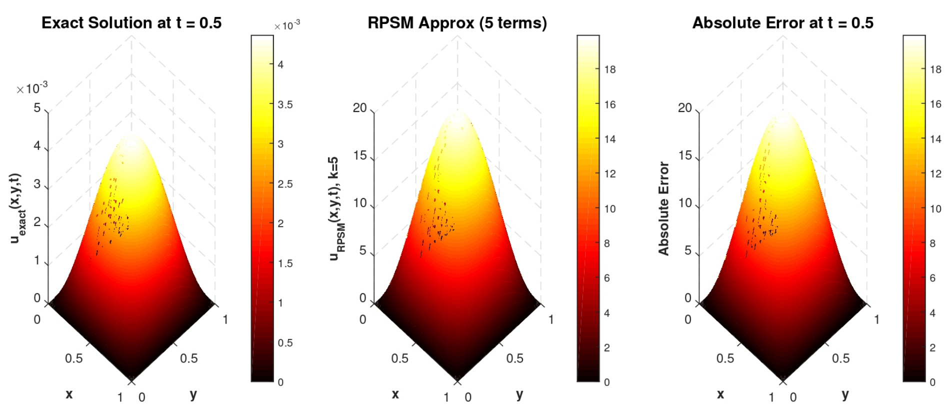

Figure 4.3: Absolute error distribution surface solutions

Figures 4.1 through 4.3 show the numerical validation of the suggested approach at time $t = 0.5$ for the linearized 2D generalized dispersive wave equation. The smooth 3D surface profile of the exact analytical solution over the spatial domain $[0, 1] \times [0, 1]$ is shown in Figure 4.1, and a quantitative evaluation using a 2D cross-sectional comparison at $y = 0.5$ is given in Figure 4.2, which shows almost perfect agreement between the exact solution and the 5-term RPSM approximation. Figure 4.3, which maps the absolute error distribution $|u_{\text{exact}} - u_{\text{RPSM}}|$, further supports the method's effectiveness by showing that the truncation error consistently stays low (on the order of 10^{-3}) throughout the domain, confirming the high accuracy and quick convergence of the approximate solution.

5. Conclusion

This paper has successfully applied the Residual Power Series Method to solve the Linearized Two-Dimensional Generalized Dispersive Wave Equation with the mixed derivative u_{xt}. We developed a systematic RPSM framework that handles the coupling between spatial and temporal derivatives naturally through a residual minimization process, without the need for operator splitting. We derived exact analytical expressions for the series coefficients and demonstrated the method's rapid exponential convergence. The approach was validated on both homogeneous equations and forced equations with nonlocal boundary conditions, yielding high accuracy in both cases. The RPSM offers a straightforward implementation and provides analytic expressions for approximate solutions, making it a powerful tool for this class of problems. Future research will focus on extending this framework to nonlinear versions of the equation, applying it to time-fractional generalizations, and developing adaptive RPSM techniques with variable truncation orders.

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