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RESEARCH ARTICLE

Sharp Bounds for Initial Toeplitz and Vandermonde Determinants Associated with the Cusp-Domain Class $\mathcal{M}_{\tan}$

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Abstract:
Let $\mathcal{A}$ denote the family of normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D}$. We investigate the cusp-domain class

$$\mathcal{M}_{\tan} = \left\{ f \in \mathcal{A} : f'(z) \prec h(z) := \frac{2(1 + \arctan z)}{1 + e^{-z}} \right\}.$$

Using an exact two-coefficient parametrization of the Schwarz function in the subordination, we obtain the sharp estimates $|\mathcal{T}_2(1)| \leq \frac{25}{16}$, $|\mathcal{T}_2(2)| \leq \frac{85}{144}$, and $|\mathcal{T}_3(1)| \leq \frac{329}{144}$, for the initial symmetric Toeplitz determinants, and $|\mathcal{V}_2(1)| \leq \frac{7}{4}$ and $|\mathcal{V}_2(2)| \leq \frac{11}{12}$, for the initial Vandermonde determinants. Each of the five determinant bounds is sharp and is attained by an explicitly constructed function in $\mathcal{M}_{\tan}$. The extremal functions are given explicitly by $F_\lambda(z) = \int_0^z h(\lambda t)\, dt$, with $\lambda = i$ for the Toeplitz bounds and $\lambda = -1$ for the Vandermonde bounds. The proofs retain the dependence between a_2 and a_3; this is essential because independent coefficient estimates do not, in general, establish sharpness of determinant functionals.

Keywords: analytic function; subordination; Schwarz function; cusp domain; Toeplitz determinant; Vandermonde determinant; sharp coefficient bound.

1. Introduction

A complex function $f(z)$ is said to be analytic at a point z_0 if it is differentiable at z_0 and every point in some neighborhood of z_0. We let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$ and normalized by

$$f(0) = 0, f'(0) = 1,$$

such that each $f \in \mathcal{A}$ has the expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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A function $f \in \mathcal{A}$ is said to be univalent if it injective in $\mathbb{D}$ and the class of all such univalent functions $f \in \mathcal{A}$ is denoted by $\mathcal{S}$. Some widely studied subclasses include Starlike function ($\mathcal{S}^*$), Convex function ($\mathcal{C}$), and close-to-convex function ($\mathcal{K}$).

A function $f \in \mathcal{A}$ is said to be starlike if it maps $\mathbb{D}$ onto domain that is starlike with respect to the origin. An analytic function w satisfying

$$w(0) = 0, \quad |w(z)| < 1, \forall z \in \mathbb{D}$$

is called a Schwarz function and is used to construct subordination relations. For $f, g \in \mathcal{A}$, the function f is said to be subordinate to g, written $f \prec g$, if there exists a Schwarz function w such that

$$f(z) = g(w(z)), \ \forall z \in \mathbb{D}.$$

In the present work, coefficient estimates are obtained directly from the Schwarz function associated with the defining subordination. An exact parametrization of its first two coefficients preserves the dependence between a_2 and a_3. This dependence is essential for deriving the sharp bounds for the initial symmetric Toeplitz and Vandermonde determinants and verifying that the proposed extremal functions attain them. Subordination is a standard mechanism for defining geometrically meaningful subclasses of analytic and univalent functions; see, for example, Duren [1], Pommerenke [2], and Ma and Minda [3].

Subclasses such as S^* and C play an important role in geometric function theory, particularly in coefficient problems and determinant inequalities. Subordination is widely used to characterize function classes and derive sharp analytic inequalities, while the Carathéodory class P serves as an important comparison class in extremal problems. These concepts provide the foundation for obtaining coefficient estimates and determinant bounds in the subclasses studied in this research.

Determinants are fundamental in linear algebra and have important applications in analysis, matrix theory, and complex analysis. Particular attention has been given to structured matrices, especially Hankel, Toeplitz, and Vandermonde matrices, whose determinants often allow explicit formulas and useful estimates.

One important tool is the Hankel determinant, which was first introduced by Pommerenke [4]. The Hankel determinant provides information about the growth and structural behavior of the coefficients of analytic functions. It was later developed by Noonan and Thomas [5] and further studied by Janteng [6], particularly in connection with the starlike and convex subclasses. The concept of the Hankel determinant and its application to the problem of estimating the coefficients of a function f have been extensively investigated in the literature; see, for example, [6–14]. Further developments concerning Hankel determinant estimates for various subclasses of univalent, starlike, convex, and related analytic functions can be found in [15–20].

On the other hand, the Toeplitz determinant is derived from a Toeplitz matrix, which is a structured matrix in which the elements on each parallel principal diagonal have the same value, see [21, 22]. Toeplitz determinants built from coefficients of analytic functions were systematically investigated by Ali, Thomas, and Vasudevarao [23]; see also Allu, Lecko, and Thomas [24]. Vandermonde determinants formed from consecutive Taylor coefficients were introduced into this line of geometric-function-theory problems by Vijayalakshmi, Bulut, and Sudharsan [25]. These two determinants encode different coefficient structures. A symmetric Toeplitz determinant uses coefficients along constant diagonals, whereas a Vandermonde determinant is the product of pairwise coefficient differences. Further developments concerning determinant-type functionals and their applications to coefficient problems for various subclasses of analytic functions can be found in [26–30]. In particular, recent studies have focused on Hankel and Toeplitz determinants involving logarithmic coefficients, inverse functions, and various subclasses of univalent and starlike functions; see [27, 29, 30]. Moreover, coefficient estimates associated with Gregory coefficients and related special functions have attracted considerable attention in recent studies; see [31–35].

The class studied here was introduced recently by El-Ityan et al. [36]. It is generated by

$$h(z) = \frac{2(1 + \arctan z)}{1 + e^{-z}}, \tag{1.2}$$

whose image is associated with a cusp-shaped domain. The corresponding class is

$$\mathcal{M}_{\tan} := \{f \in \mathcal{A} : f'(z) \prec h(z)\}. \tag{1.3}$$

The initial coefficient estimates and several Hankel-type functionals for $\mathcal{M}_{\tan}$ were considered in [36]. The purpose of the present work is different: we study symmetric Toeplitz determinants and Vandermonde determinants formed from the Taylor coefficients in (1.1).

Our main contribution is five best-possible constants: three for the initial symmetric Toeplitz determinants and two for the initial Vandermonde determinants. The adjective initial in the title is deliberate. The results concern $\mathcal{T}_2(1)$, $\mathcal{T}_2(2)$, $\mathcal{T}_3(1)$, $\mathcal{V}_2(1)$, and $\mathcal{V}_2(2)$; higher indices require additional coefficient region information and are not claimed here. The proofs are based on an exact admissible region for (a_2, a_3) rather than on separate estimates for $|a_2|$ and $|a_3|$. This also makes the equality cases transparent and prevents a common logical error: multiplying several individually sharp coefficient bounds need not produce a sharp bound for a determinant because the equality cases may be incompatible.

2. Preliminaries

We first fix the determinant conventions. This is necessary because "Toeplitz determinant" is also used for a Hermitian matrix in parts of the literature. Following Ali et al. [23] and Allu et al. [24, Definition 1.3], we adopt the following symmetric Toeplitz determinant convention.

Definition 2.1 (Symmetric Toeplitz determinant)**.** For $f \in \mathcal{A}$ of the form (1.1), $q \geq 1$, and $n \geq 1$, set

$$\mathcal{T}_q(n; f) := \det\left[a_{n+|j-k|}\right]_{j,k=1}^{q}. \tag{2.4}$$

In particular,

$$\mathcal{T}_2(1; f) = \begin{vmatrix} 1 & a_2 \\ a_2 & 1 \end{vmatrix} = 1 - a_2^2, \tag{2.5}$$

$$\mathcal{T}_2(2; f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix} = a_2^2 - a_3^2, \tag{2.6}$$

$$\mathcal{T}_3(1; f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} = 1 - 2a_2^2 + 2a_2^2 a_3 - a_3^2. \tag{2.7}$$

Following Vijayalakshmi, Bulut, and Sudharsan [25], we define the Vandermonde determinant generated by consecutive Taylor coefficients as follows.

Definition 2.2 (Vandermonde determinant)**.** For $f \in \mathcal{A}$, $q \geq 1$, and $n \geq 1$, define

$$\mathcal{V}_q(n; f) := \det\left[a_{n+j-1}^{k-1}\right]_{j,k=1}^{q} = \prod_{n \leq j < k \leq n+q-1} (a_k - a_j). \tag{2.8}$$

Thus

$$\mathcal{V}_2(1; f) = a_2 - a_1 = a_2 - 1, \qquad \mathcal{V}_2(2; f) = a_3 - a_2. \tag{2.9}$$

From equation (1.1), consider the analytic function

$$h(z) := \frac{2(1+\arctan z)}{1+e^{-z}}, \qquad z \in \mathbb{D}.$$

The image domain $h(\mathbb{D})$ is referred to as the cusp domain associated with h.

Definition 2.3 ([36]). The subclass $\mathcal{M}_{\tan}$ of $\mathcal{A}$ associated with the cusp domain $h(\mathbb{D})$ is defined by

$$\mathcal{M}_{\tan} := \left\{ f \in \mathcal{A} : f'(z) \prec h(z) = \frac{2(1+\arctan z)}{1+e^{-z}}, \quad z \in \mathbb{D} \right\}.$$

Equivalently, a function $f \in \mathcal{A}$ belongs to $\mathcal{M}_{\tan}$ if and only if there exists a Schwarz function ω, analytic in $\mathbb{D}$ and satisfying

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1,$$

such that

$$f'(z) = h(\omega(z)) = \frac{2\big(1+\arctan(\omega(z))\big)}{1+e^{-\omega(z)}}, \qquad z \in \mathbb{D}.$$

The function h has the Maclaurin expansion

$$h(z) = 1 + \frac{3}{2}z + \frac{1}{2}z^2 - \frac{3}{8}z^3 - \frac{5}{24}z^4 + \frac{49}{240}z^5 + \frac{17}{144}z^6 + \cdots.$$

In particular,

$$h(0) = 1,$$

which is consistent with the normalization $f'(0) = 1$ for $f \in \mathcal{A}$.

3. Main Results

We begin by recording the Schwarz-function parametrization that will be used throughout the proofs of the main results.

Lemma 3.1 (Two-coefficient Schwarz parametrization). *Let*

$$w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

be a Schwarz function. Then

$$|c_1| \le 1, \qquad c_2 = (1-|c_1|^2)\xi \tag{3.10}$$

for some $\xi \in \overline{\mathbb{D}}$. Conversely, for every $c_1, \xi \in \overline{\mathbb{D}}$, there exists a Schwarz function whose first two coefficients satisfy (3.10).

Proof. By Schwarz's lemma, $|c_1| = |w'(0)| \le 1$. We first separate the equality case. If $|c_1| = 1$, the equality statement in Schwarz's lemma yields $w(z) = c_1 z$ throughout $\mathbb{D}$. Hence $c_2 = 0$, and (3.10) holds for any $\xi \in \overline{\mathbb{D}}$. Suppose now that $|c_1| < 1$. The function $\varphi(z) := \frac{w(z)}{z}$ extends analytically to $\mathbb{D}$ and has the expansion

$$\varphi(z) = c_1 + c_2 z + \cdots.$$

Schwarz's lemma gives $|\varphi(z)| \le 1$. In fact, $|\varphi(z)| < 1$ for every $z \in \mathbb{D}$, otherwise, the maximum-modulus principle would make φ a unimodular constant, contradicting $|\varphi(0)| = |c_1| < 1$. Thus $\varphi : \mathbb{D} \to \mathbb{D}$, and the Schwarz–Pick inequality may legitimately be applied at the origin. It gives

$$|c_2| = |\varphi'(0)| \le 1 - |\varphi(0)|^2 = 1 - |c_1|^2.$$

Defining $\xi = \frac{c_2}{(1-|c_1|^2)}$ proves (3.10) in the remaining case. Conversely, let $c_1, \xi \in \overline{\mathbb{D}}$. If $|c_1| < 1$, define

$$\varphi(z) = \frac{c_1 + \xi z}{1 + \overline{c_1}\xi z}, \qquad w(z) = z\varphi(z).$$

The first function is the composition of the disk map $z \mapsto \xi z$ and the disk automorphism $\zeta \mapsto \frac{(c_1+\zeta)}{(1+\overline{c_1}\zeta)}$. Consequently, w is a Schwarz function. Moreover,

$$\varphi(z) = c_1 + (1 - |c_1|^2)\xi z + O(z^2),$$

and hence

$$w(z) = c_1 z + (1 - |c_1|^2)\xi z^2 + O(z^3).$$

If $|c_1| = 1$, take $w(z) = c_1 z$; then the prescribed second coefficient is $(1 - |c_1|^2)\xi = 0$. This establishes the converse and completes the proof. □

Lemma 3.2 (Exact initial coefficient representation)**.** *Let $f \in \mathcal{M}_{\tan}$ have the expansion (1.1). There are $c_1 \in \overline{\mathbb{D}}$ and $\xi \in \overline{\mathbb{D}}$ such that*

$$a_2 = \frac{3}{4}c_1, \tag{3.11}$$

$$a_3 = \frac{1}{6}c_1^2 + \frac{1}{2}(1 - |c_1|^2)\xi. \tag{3.12}$$

Consequently, if $r = |c_1|$, then

$$|a_2| = \frac{3}{4}r, \qquad |a_3| \le \frac{1}{2} - \frac{1}{3}r^2. \tag{3.13}$$

For each fixed $r \in [0, 1]$, the bound for $|a_3|$ in (3.13) is sharp and is attained by choosing $c_1 = r$ and $\xi = 1$. Both formulas in (3.11)–(3.12) retain the dependence between a_2 and a_3.

Proof. Because $f \in \mathcal{M}_{\tan}$, there exists a Schwarz function $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ such that

$$f'(z) = h(w(z)). \tag{3.14}$$

Using (1.2), the right-hand side has the expansion

$$\begin{aligned} h(w(z)) &= 1 + \frac{3}{2}w(z) + \frac{1}{2}w(z)^2 - \frac{3}{8}w(z)^3 + \cdots \\ &= 1 + \frac{3}{2}c_1 z + \left(\frac{3}{2}c_2 + \frac{1}{2}c_1^2\right) z^2 + \left(\frac{3}{2}c_3 + c_1 c_2 - \frac{3}{8}c_1^3\right) z^3 + \cdots . \end{aligned}$$

On the other hand,

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots .$$

Comparison of the coefficients of z and z^2 yields

$$2a_2 = \frac{3}{2}c_1, \qquad 3a_3 = \frac{3}{2}c_2 + \frac{1}{2}c_1^2,$$

which gives

$$a_2 = \frac{3}{4}c_1, \qquad a_3 = \frac{1}{2}c_2 + \frac{1}{6}c_1^2.$$

Substitution of $c_2 = (1 - |c_1|^2)\xi$ from Lemma 3.1 proves (3.11) and (3.12). Finally,

$$|a_3| \le \frac{1}{6}|c_1|^2 + \frac{1}{2}(1 - |c_1|^2)|\xi| \le \frac{1}{6}r^2 + \frac{1}{2}(1 - r^2) = \frac{1}{2} - \frac{1}{3}r^2,$$

which is (3.13). To verify sharpness for each fixed $r \in [0,1]$, choose $c_1 = r$ and $\xi = 1$. By Lemma 3.1, these parameters correspond to an admissible Schwarz function and hence, through $f'(z) = h(w(z))$ with $f(0) = 0$, to a function $f \in \mathcal{M}_{\tan}$. Substitution into (3.12) gives

$$a_3 = \frac{r^2}{6} + \frac{1 - r^2}{2} = \frac{1}{2} - \frac{r^2}{3} \geq \frac{1}{6} > 0.$$

Therefore,

$$|a_3| = \frac{1}{2} - \frac{r^2}{3},$$

which proves that the bound in (3.13) is sharp for every $r \in [0,1]$. □

The following one-parameter family will provide every equality case in the main results.

Proposition 3.1 (Extremal rotations). *For $|\lambda| = 1$, define*

$$F_\lambda(z) := \int_0^z h(\lambda t)\, dt. \tag{3.15}$$

Then $F_\lambda \in \mathcal{M}_{\tan}$ and

$$F_\lambda(z) = z + \frac{3}{4}\lambda z^2 + \frac{1}{6}\lambda^2 z^3 - \frac{3}{32}\lambda^3 z^4 - \frac{1}{24}\lambda^4 z^5 + \cdots. \tag{3.16}$$

Proof. The function $w(z) = \lambda z$ is a Schwarz function, and $F'_\lambda(z) = h(\lambda z) = h(w(z))$. Hence $F_\lambda \in \mathcal{M}_{\tan}$ by (1.3). Integrating the series in (1.2) term by term gives (3.16). □

3.1. Sharp Toeplitz determinant bounds

We begin with the two initial determinants of order two.

Theorem 3.1. *If $f \in \mathcal{M}_{\tan}$, then*

$$|\mathcal{T}_2(1; f)| \leq \frac{25}{16}, \tag{3.17}$$

$$|\mathcal{T}_2(2; f)| \leq \frac{85}{144}. \tag{3.18}$$

Both inequalities are sharp. Equality in each is attained by F_i and F_{-i} from (3.15).

Proof. Let c_1 and ξ be as in Lemma 3.2, and put $r = |c_1| \in [0,1]$. For (3.17), equations (2.5) and (3.11) give

$$\begin{aligned} |\mathcal{T}_2(1; f)| &= |1 - a_2^2| \\ &\leq 1 + |a_2|^2 \\ &= 1 + \frac{9}{16} r^2 \\ &\leq 1 + \frac{9}{16} = \frac{25}{16}. \end{aligned}$$

For (3.18), use (2.6) and (3.13) to obtain

$$\begin{aligned} |\mathcal{T}_2(2; f)| &= |a_2^2 - a_3^2| \\ &\leq |a_2|^2 + |a_3|^2 \\ &\leq \frac{9}{16} r^2 + \left(\frac{1}{2} - \frac{1}{3} r^2\right)^2. \end{aligned}$$

Set $t = r^2 \in [0,1]$. The last expression is

$$\Phi(t) = \frac{1}{4} + \frac{11}{48}t + \frac{1}{9}t^2. \tag{3.19}$$

Since

$$\Phi'(t) = \frac{11}{48} + \frac{2}{9}t > 0 \qquad (0 \le t \le 1),$$

Φ is strictly increasing on $[0,1]$. Consequently,

$$|\mathcal{T}_2(2;f)| \le \Phi(1) = \frac{9}{16} + \frac{1}{36} = \frac{85}{144}.$$

It remains to verify sharpness, including compatibility of the equality conditions. For $f = F_i$, Proposition 3.1 gives

$$a_2 = \frac{3i}{4}, \qquad a_3 = -\frac{1}{6}.$$

Therefore

$$\mathcal{T}_2(1;F_i) = 1 - \left(\frac{3i}{4}\right)^2 = \frac{25}{16}$$

and

$$\mathcal{T}_2(2;F_i) = \left(\frac{3i}{4}\right)^2 - \left(-\frac{1}{6}\right)^2 = -\frac{9}{16} - \frac{1}{36} = -\frac{85}{144}.$$

Thus the moduli equal the right-hand sides of (3.17) and (3.18). The choice $\lambda = -i$ gives the same moduli. Hence both constants are best possible. □

We now treat the first determinant of order three. A direct triangle inequality applied to the four terms in (2.7) does not give the sharp constant. The factorization in the proof below is the key point.

Theorem 3.2. *If $f \in \mathcal{M}_{\tan}$, then*

$$|\mathcal{T}_3(1;f)| \le \frac{329}{144}. \tag{3.20}$$

The inequality is sharp, with equality for F_i and F_{-i}.

Proof. From (2.7),

$$\mathcal{T}_3(1;f) = (1 - a_3)(1 + a_3 - 2a_2^2). \tag{3.21}$$

Let

$$u = c_1^2, \qquad t = |u| = |c_1|^2 \in [0,1], \qquad R = \frac{1-t}{2}.$$

By Lemma 3.2,

$$a_2^2 = \frac{9}{16}u, \qquad a_3 = \frac{1}{6}u + R\xi, \qquad |\xi| \le 1. \tag{3.22}$$

Substitution of (3.22) into (3.21) gives

$$\mathcal{T}_3(1;f) = \left(1 - \frac{1}{6}u - R\xi\right)\left(1 - \frac{23}{24}u + R\xi\right) = \left(1 - \frac{1}{6}u\right)\left(1 - \frac{23}{24}u\right) + \frac{19}{24}Ru\xi - R^2\xi^2. \tag{3.23}$$

Indeed, the coefficient of $R\xi$ in the product is

$$\left(1 - \frac{1}{6}u\right) - \left(1 - \frac{23}{24}u\right) = \frac{19}{24}u.$$

Taking absolute values in (3.23), using $|u| = t$, $|\xi| \le 1$, and $R \ge 0$, we obtain

$$\begin{aligned}|\mathcal{T}_3(1; f)| &\le \left|1 - \frac{1}{6}u\right| \left|1 - \frac{23}{24}u\right| + \frac{19}{24}Rt + R^2 \\ &\le \left(1 + \frac{1}{6}t\right)\left(1 + \frac{23}{24}t\right) + \frac{19}{24}Rt + R^2.\end{aligned}$$

Since $R = \frac{1-t}{2}$, the last expression reduces exactly to

$$\Psi(t) = \left(1 + \frac{1}{6}t\right)\left(1 + \frac{23}{24}t\right) + \frac{19}{48}t(1-t) + \frac{1}{4}(1-t)^2\frac{5}{4} + \frac{49}{48}t + \frac{1}{72}t^2. \tag{3.24}$$

Now

$$\Psi'(t) = \frac{49}{48} + \frac{1}{36}t > 0 \qquad (0 \le t \le 1),$$

so Ψ is increasing. Hence

$$|\mathcal{T}_3(1; f)| \le \Psi(1) = \frac{5}{4} + \frac{49}{48} + \frac{1}{72} = \frac{329}{144}.$$

To verify sharpness, consider $f = F_i$, whose associated Schwarz function is $w(z) = iz$. Thus

$$a_2 = \frac{3i}{4}, \qquad a_3 = -\frac{1}{6}, \qquad u = -1, \qquad t = 1, \qquad R = 0.$$

Since $R = 0$, all terms involving R vanish. Moreover,

$$\left|1 - \frac{u}{6}\right| = 1 + \frac{t}{6}, \qquad \left|1 - \frac{23u}{24}\right| = 1 + \frac{23t}{24}.$$

onsequently, every inequality in the preceding estimate becomes an equality. Direct substitution into (3.21) gives

$$T_3(1; F_i) = \left(1 + \frac{1}{6}\right)\left(1 - \frac{1}{6} - 2\left(\frac{3i}{4}\right)^2\right) = \frac{7}{6} \cdot \frac{47}{24} = \frac{329}{144}.$$

Therefore, the bound in (3.20) is attained and is sharp. The same conclusion holds for F_{-i}, since its coefficients satisfy $a_2 = -\frac{3i}{4}$ and $a_3 = -\frac{1}{6}$. □

3.2. Sharp Vandermonde determinant bounds

The Vandermonde determinant of order two is a difference of consecutive Taylor coefficients. Although the formulas are shorter than in the Toeplitz case, the coefficient dependence in Lemma 3.2 is again needed to obtain the exact bound for $a_3 - a_2$.

Theorem 3.3. *If $f \in \mathcal{M}_{\tan}$, then*

$$|\mathcal{V}_2(1; f)| \le \frac{7}{4}, \tag{3.25}$$

$$|\mathcal{V}_2(2; f)| \le \frac{11}{12}. \tag{3.26}$$

Both inequalities are sharp. Equality in both is attained by F_{-1}.

Proof. Let c_1, ξ, and $r = |c_1|$ be as in Lemma 3.2. From (2.9) and (3.11),

$$\begin{aligned} |\mathcal{V}_2(1; f)| &= |a_2 - 1| \\ &= \left|\frac{3}{4}c_1 - 1\right| \\ &\le 1 + \frac{3}{4}r \\ &\le \frac{7}{4}, \end{aligned}$$

which proves (3.25). For the second determinant, equations (3.11) and (3.12) yield

$$\begin{aligned} |\mathcal{V}_2(2; f)| &= |a_3 - a_2| \\ &= \left|\frac{1}{6}c_1^2 + \frac{1}{2}(1 - r^2)\xi - \frac{3}{4}c_1\right| \\ &\le \frac{1}{6}r^2 + \frac{1}{2}(1 - r^2) + \frac{3}{4}r \\ &= \frac{1}{2} + \frac{3}{4}r - \frac{1}{3}r^2 =: G(r). \end{aligned}$$

For $0 \le r \le 1$,

$$G'(r) = \frac{3}{4} - \frac{2}{3}r \ge \frac{1}{12} > 0.$$

Thus G is strictly increasing on $[0, 1]$, and

$$|\mathcal{V}_2(2; f)| \le G(1) = \frac{1}{2} + \frac{3}{4} - \frac{1}{3} = \frac{11}{12}.$$

Finally, Proposition 3.1 gives, for $f = F_{-1}$,

$$a_2 = -\frac{3}{4}, \qquad a_3 = \frac{1}{6}.$$

Therefore

$$|\mathcal{V}_2(1; F_{-1})| = \left|-\frac{3}{4} - 1\right| = \frac{7}{4}$$

and

$$|\mathcal{V}_2(2; F_{-1})| = \left|\frac{1}{6} + \frac{3}{4}\right| = \frac{11}{12}.$$

This proves the sharpness of both estimates. □

For completeness, the same coefficient region immediately gives another sharp initial gap that is one factor in the third-order Vandermonde determinant.

Corollary 3.1. *If $f \in \mathcal{M}_{\tan}$, then*

$$|a_3 - a_1| = |a_3 - 1| \le \frac{3}{2}. \tag{3.27}$$

The bound is sharp and is attained by

$$G(z) := \int_0^z h(-t^2)\, dt = z - \frac{1}{2}z^3 + \cdots.$$

Proof. By (3.13), $|a_3| \le \frac{1}{2}$, and hence

$$|a_3 - 1| \le 1 + |a_3| \le \frac{3}{2}.$$

For $w(z) = -z^2$, we have w Schwarz and $G'(z) = h(-z^2)$, so $G \in \mathcal{M}_{\tan}$. Since $a_3 = -\frac{1}{2}$ for this function, $|a_3 - 1| = \frac{3}{2}$. □

The results can be summarized as follows.

Functional	Sharp bound	Extremal choice
$\lvert\mathcal{T}_2(1;f)\rvert$	$\frac{25}{16}$	F_i or F_{-i}
$\lvert\mathcal{T}_2(2;f)\rvert$	$\frac{85}{144}$	F_i or F_{-i}
$\lvert\mathcal{T}_3(1;f)\rvert$	$\frac{329}{144}$	F_i or F_{-i}
$\lvert\mathcal{V}_2(1;f)\rvert$	$\frac{7}{4}$	F_{-1}
$\lvert\mathcal{V}_2(2;f)\rvert$	$\frac{11}{12}$	F_{-1}

Remark 1 (Why separate coefficient bounds are insufficient). The coefficient bounds

$$|a_2| \le \frac{3}{4}, \qquad |a_3| \le \frac{1}{2}$$

are individually sharp, but equality cannot hold simultaneously in both bounds. Indeed, Lemma 3.2 gives

$$|a_2| = \frac{3}{4}r, \qquad |a_3| \le \frac{1}{2} - \frac{r^2}{3}, \qquad 0 \le r \le 1.$$

If $|a_2| = \frac{3}{4}$, then $r = 1$, and hence

$$|a_3| \le \frac{1}{6} < \frac{1}{2}.$$

Conversely, if $|a_3| = \frac{1}{2}$, then

$$\frac{1}{2} \le \frac{1}{2} - \frac{r^2}{3},$$

which forces $r = 0$ and therefore $a_2 = 0$. Thus, no function in $\mathcal{M}_{\tan}$ attains equality in both coefficient bounds simultaneously. Consequently, using these coefficient bounds separately may produce a valid upper bound for a determinant, but the sharpness of that bound cannot be inferred solely from the individual sharpness of the coefficient estimates. The dependence between a_2 and a_3, as expressed in Lemma 3.2, must also be taken into account.

Remark 2 (Symmetric versus Hermitian Toeplitz determinants). The determinants in Definition 2.1 contain a_j symmetrically and do not contain conjugates. A Hermitian–Toeplitz determinant instead uses $\overline{a_j}$ below the main diagonal. Its determinant is a different functional and the constants proved here must not be transferred to that setting without a separate proof.

4. Conclusion

We established sharp bounds for the first two symmetric Toeplitz determinants of order two, the initial symmetric Toeplitz determinant of order three, and the first two Vandermonde determinants for the cusp-domain class $\mathcal{M}_{\tan}$. The extremals are rotations of the cusp generator integrated once, and all equality cases are verified directly. Several natural problems remain open. The next targets are the exact bounds for $\mathcal{T}_2(3)$ and $\mathcal{T}_3(2)$, which involve a_4, and the sharp third-order Vandermonde determinant

$$\mathcal{V}_3(1;f) = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2).$$

The last problem cannot be solved sharply by multiplying the three separate gap bounds, because their extremal Schwarz functions are different. A sharp solution requires a simultaneous optimization over the exact coefficient body, or an equivalent Schur-parameter representation. Hermitian–Toeplitz determinants, inverse coefficients, and logarithmic coefficients of $\mathcal{M}_{\tan}$ provide further independent directions.

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